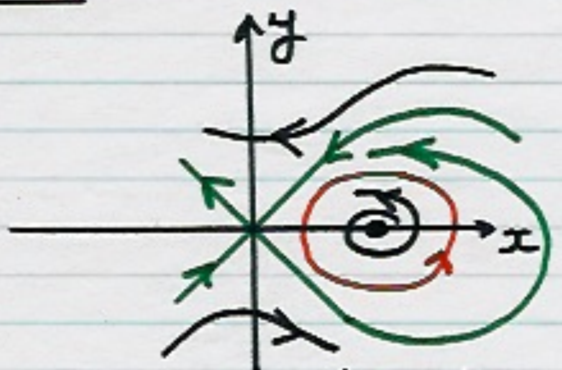
Example of Homoclinic Bifurcation of a limit cycle.

Global bifurcation of limit cycles & phase plane surgery

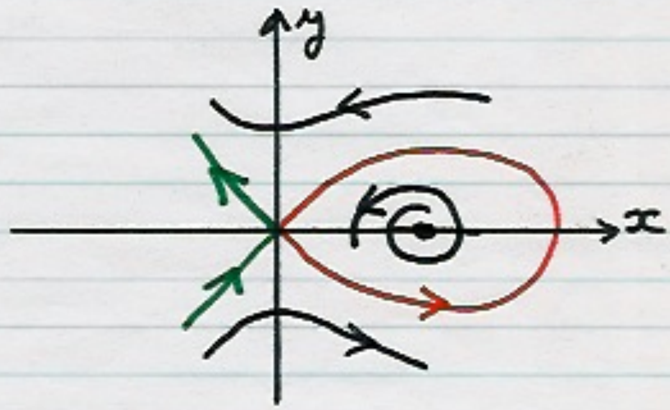
→ Homoclinic bifurcation: Saddle moves onto limit cycle and destroys it (at bifurcation [redacted] limit cycle will become an homoclinic conection for saddle

+ Calculation of period "near bif."

Idea

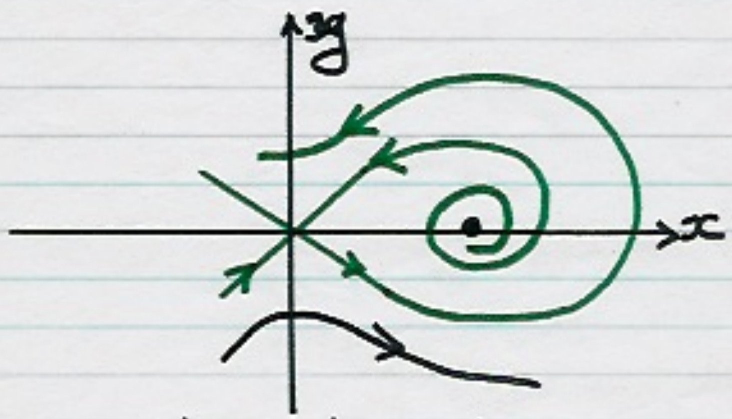


Before bifurcation



at bifurcation

(Note cycle graph)



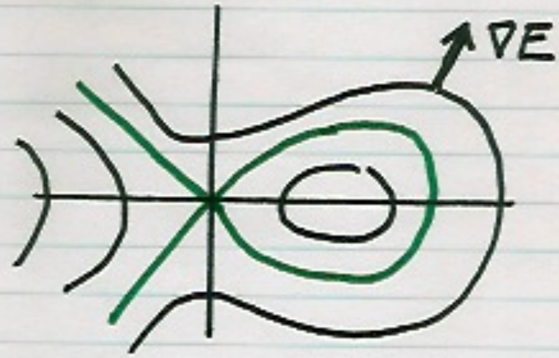
After bifurcation

T. Contents

1. Example by phase plane surgery.
2. Show period limit cycle $\sim -\ln \mu$ (μ = bifurcation distance)

Example by phase plane surgery

1) Construct "energy" with level curves as in figure



Example

$$E = \frac{1}{2}y^2 - \frac{1}{2}x^2 + \frac{1}{3}x^3$$

(Surface = "lake in crater with broken side")

Minimum at $\begin{cases} y=0 \quad x=1 \\ E = -1/6 \end{cases}$

saddle at $\begin{cases} x=y=0 \\ E = 0 \end{cases}$

Now write

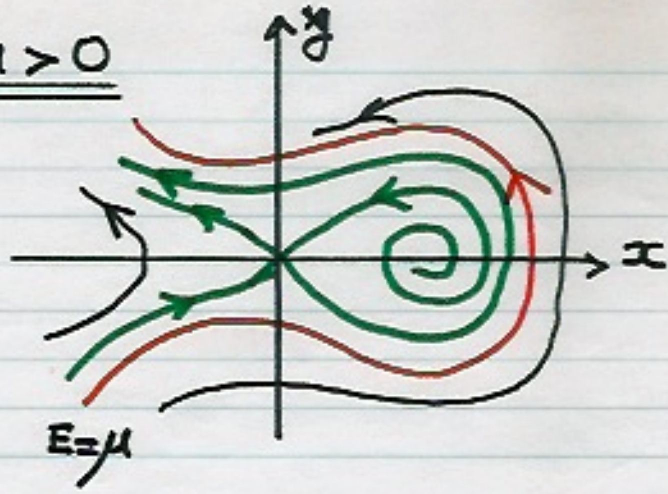
$$\left.\begin{aligned} \dot{x} &= a E_x - b E_y \\ \dot{y} &= b E_x + a E_y \end{aligned}\right\} \Rightarrow \dot{E} = a(\nabla E)^2$$

Let now $\left.\begin{aligned} & a = -(E-\mu) = \mu - E \\ & \mu > -1/6 \end{aligned}\right\}$ $E = \mu$ is invariant-attracting curve!

$b = 1 \Longrightarrow \begin{cases} \text{counterclockwise flow} \\ \text{around minimum} \end{cases}$

Then homoclinic bifurcation occurs at $\mu = 0$.

Picture for $\mu > 0$



Note book example: E = same as here!

$$\dot{x} = y = E_y$$
$$\dot{y} = x - x^2 + (\mu + x)y = -E_x + (\mu + x)E_y$$

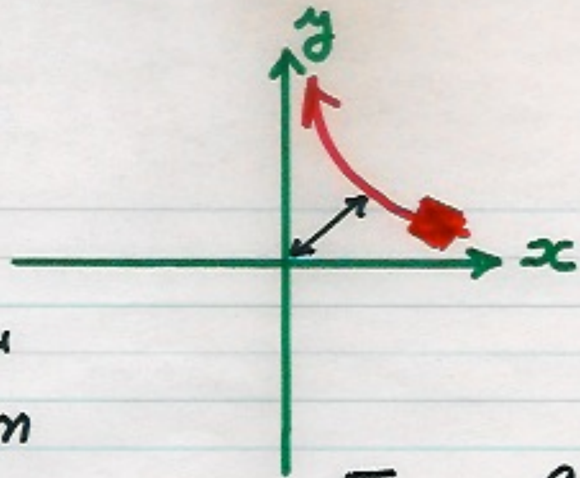
$\therefore \quad \dot{E} = (\mu + x) E_y^2$ — Works more-or-less same way, but harder to do any analysis

Period Limit cycle: Goes to ∞ like time it takes to go by a saddle!

We claim this is $\ln \mu$, where μ = distance to critical value.

How is this?

Consider a typical saddle and an orbit whose closest approach to the saddle in the phase



plane is $0 < \mu << 1$

Since μ is small we can linearize and the problem reduces to

$$\left.\begin{aligned} \dot{x} &= -ax \\ \dot{y} &= by \end{aligned}\right\} \qquad a, b > 0$$

$$x = \mu \frac{\sqrt{b}}{\sqrt{a+b}}\, e^{-a(t-t_0)}$$

$$y = \mu \frac{\sqrt{a}}{\sqrt{a+b}}\, e^{b(t-t_0)}$$

[redacted]

Note $x^2 + y^2 = \mu^2 \left\{ \frac{b}{a+b} e^{-a\tau} + \frac{a}{a+b} e^{b\tau} \right\}$, $\tau = t - t_0$

has its minimum at $\tau = 0$, and the minimum is μ^2

[redacted]

So, if we now ask: How long does it take for this orbit to go from $\{x = O(1),\ y \text{ small}\}$ to $\{x \text{ small},\ y = O(1)\}$? The answer is:

From $e^{-a\tau} = O(\mu^{-1})$ i.e. $\tau = O(a^{-1} \ln \mu)$

to $e^{b\tau} = O(\mu^{-1})$ i.e. $\tau = O(-b^{-1} \ln \mu)$

That is, it takes $\underline{\Delta t = O(|\ln \mu|)}$