Finish up Hopf
and start with
"Global Bifurcations"

cf Ref. A is the linearize matrix

$$\boxed{A\vec{v} = i\omega\vec{v}} \qquad \omega > 0$$

$$\boxed{\vec{v} = \vec{v}_1 - i\vec{v}_2}$$

$$\boxed{\begin{aligned} A\vec{v}_1 &= \omega\vec{v}_2 \\ A\vec{v}_2 &= -\omega\vec{v}_1 \end{aligned}}$$

Then can write any vector $\boxed{\vec{Y} = x v_1 + y v_2}$

$$\boxed{z = x + iy}$$

AY $\Longleftrightarrow$ $z \to i\omega z$

$$x = (z+\bar{z})/2$$
$$y = (z-\bar{z})/2i$$

$$\boxed{\dot{z} = f(z,\bar{z},\delta)}$$

Assume C.P. in $z=0$

and Bifurcation happens at $\delta = 0$

Expand for z small and δ small

$$\dot{z} = i\omega z + \delta(az+b\bar{z}) + \cancel{\text{quadratic in } z}$$
$$+ (cz^3 + dz^2\bar{z} + ez\bar{z}^2 + fz^3)$$
$$+ O(\delta^2, \delta z^2, z^4)$$

No quadratic terms
Simplifying assumption

Let $0 < \epsilon \ll 1$ $\quad \epsilon = \sqrt{|\delta|}$

Then take $z = O(\epsilon)$

$$z = \epsilon z_0(t, T) + \epsilon^3 z_2(t, T) + \cdots$$

$T = \epsilon^2 t$ $\quad$ no z_1 because no quadratic terms

$O(\epsilon)$ $\quad \boxed{z_{0t} = i\omega z_0}$

$$z_0 = A(T) e^{i\omega t}$$

$O(\epsilon^3)$ $\quad \boxed{\sigma = \operatorname{sign}(\delta)}$

$$z_{2t} = i\omega z_2 +$$

$$\frac{1}{z_0 T} + \sigma(a z_0 + b\bar{z}_0)$$

$$+ (c z_0^3 + d z_0^2 \bar{z}_0 + e z_0 \bar{z}_0^2 + f \bar{z}_0^3)$$

$$z_{2t} - i\omega z_2 = \left(-A_T + \sigma \hat{a} A + d|A|^2 A\right) e^{i\omega t}$$

$$+ (\ )e^{-i\omega t} + (\ )e^{3i\omega t} + (\ )e^{-i3\omega t}$$

Eliminate Res.

$$\boxed{A_T = \sigma a A + d|A|^2 A}$$

$$A = \rho e^{i\varphi}$$

$$\frac{d}{dT}\rho = \sigma \operatorname{Re}(a)\rho + \operatorname{Re}(d)\rho^3$$

$$\frac{d}{dT}\varphi = \sigma \operatorname{Im}(a) + \operatorname{Im}(d)\rho^2$$

$$\boxed{\begin{aligned} &Z_{2t} - i\omega Z_2 = e^{-i\omega t} \\ &Z_2 = \alpha e^{-i\omega t} + \beta e^{i\omega t} \\ &2i\omega\alpha = 1 \qquad \boxed{\alpha = \frac{-1}{2i\omega}} \end{aligned}}$$

#1 $\mathrm{Re}(a) \neq 0$

#2 Generic Asymptotic
$\mathrm{Re}(d) \neq 0$

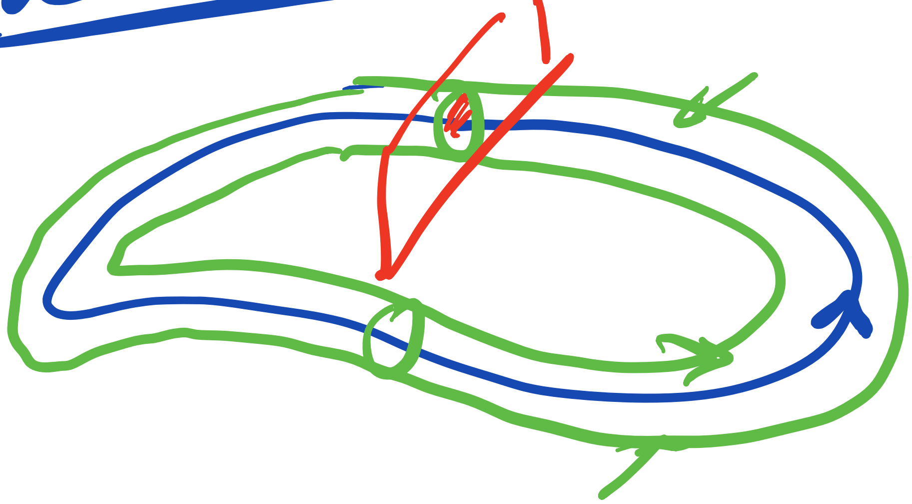
$\therefore$ can scale so the
an near to you

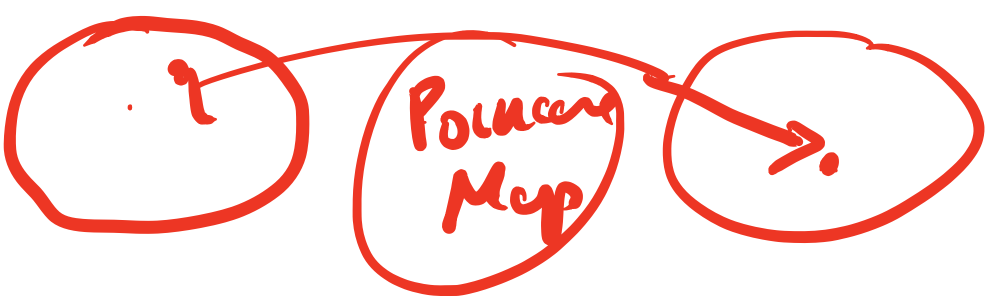
$$\frac{d}{dE}\tilde{p} = \pm\tilde{p} + \tilde{p}^3$$

$$\boxed{\frac{d}{dt}\tilde{\rho} = \pm\tilde{\rho} \pm \tilde{\rho}^3}$$

Global Bifurcation

limit cycle





First assume Poincare map ||

persists

No C.P. in paper

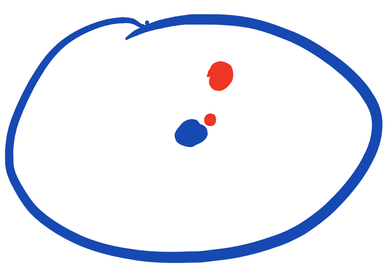
$$\text{Disk} \xrightarrow{P} \text{Disk}$$

Limit cycle = fixed point of P

$$\boxed{P(x) = x}$$

linearize near L. limit

$$P(x) \simeq A(x - x_0)$$



Stable

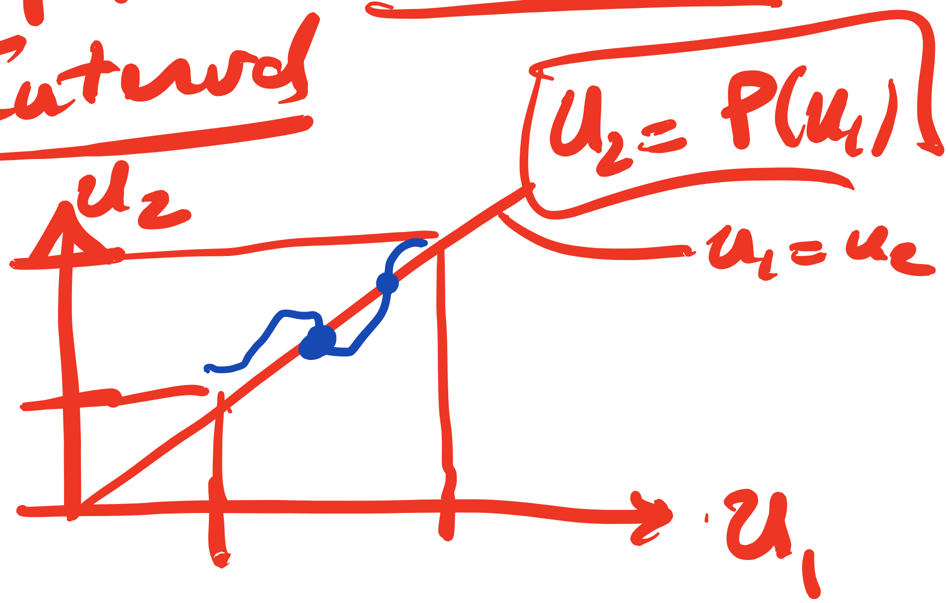
Eigenvalues

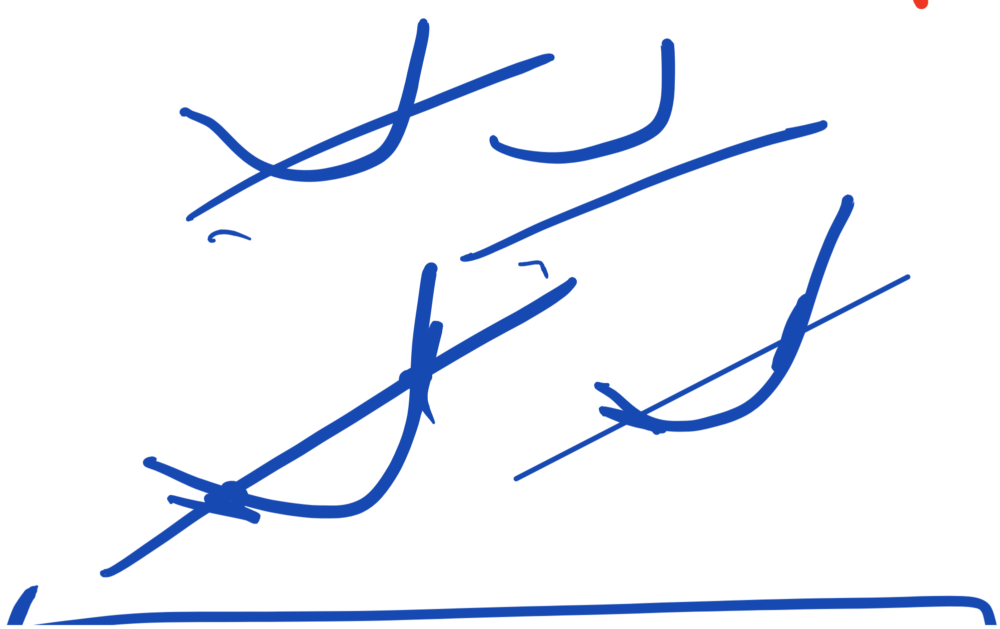
A are $|\lambda| < 1$

Case 1   1 eigenvalue crosses $|\lambda| = 1$

The problem becomes 1-D

Map from Interval to Interval

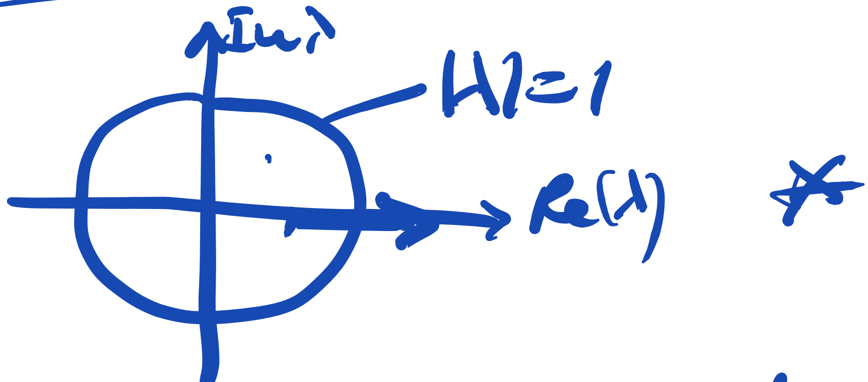


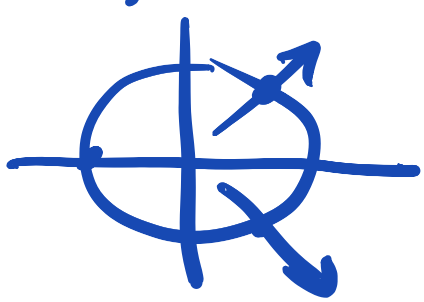


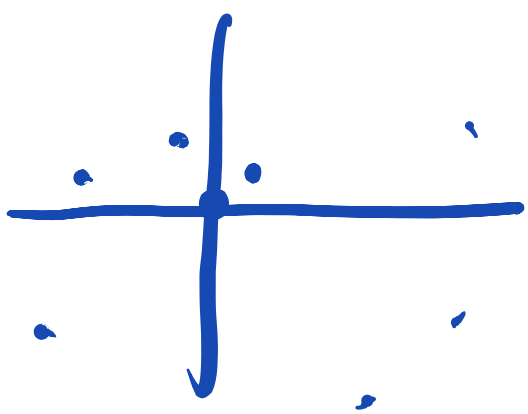
Saddle node Bif. of limit cycle

Hopf: period $O(1)$
amplitude $O(\sqrt{\delta})$

Saddle Node period $O(1)$
amplitude $O(1)$

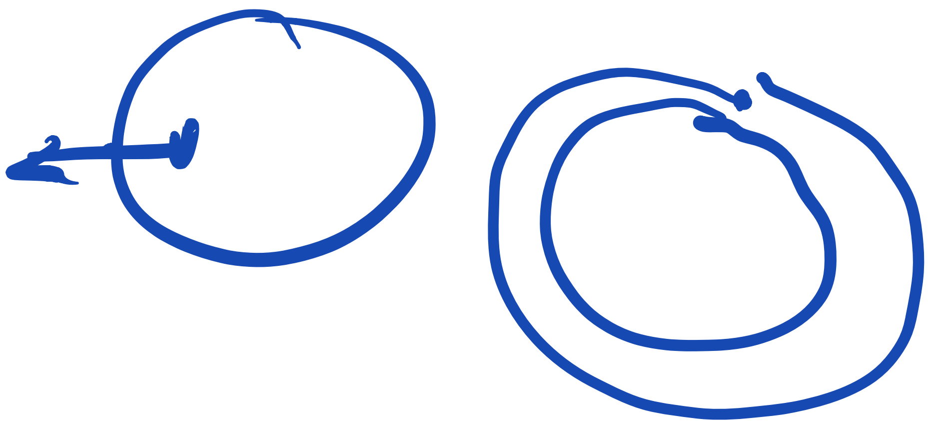






Hopf of limit cycles

2 period sols



Next step what if a C.P. shows up

$\Rightarrow$ C.P. is on limit cycle at bifurcation

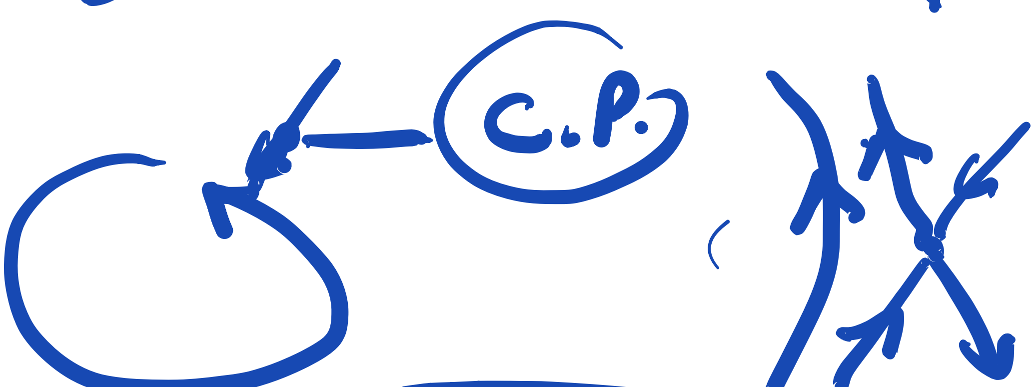
• Limit cycle destroyed

before bifurcation

Infinite period bif.

Amplitude at birth $= O(1)$

period at birth = Infinite !



Homoclinic Ref. of
C.P.