Example: Hopf bifurcation in the plane 1

using complex notation.

Consider a system $\dot{x} = f_p(x,y,\delta)$, $\dot{y} = g_p(x,y,\delta)$ depending on a parameter δ. Assume now that the origin is an isolated critical point that switches between stable and unstable spiral as δ crosses $\delta = 0$. We also assume that f_p and g_p are smooth functions of (x, y, δ).

Let $\underline{\underline{A_0}}$ be the matrix for the linearized system near the origin for $r = 0$. (1)

Then A_0 has the eigenvalues $\lambda = \pm i\omega$, where $\omega > 0$.

Let $\vec{v} = \vec{v}_1 - i\vec{v}_2$ be an eigenvector corresponding to $i\omega$, so that $A_0\vec{v} = i\omega\vec{v}$ (2)

Thus $A_0\vec{v}_1 = \omega\vec{v}_2$ and $A_0\vec{v}_2 = -\omega\vec{v}_1$

<u>Note 1</u> : Neither $\vec{V}_1$ nor $\vec{V}_2$ can vanish, because if one does, both do (use the eqns. in (2)) and $\vec{V} \neq 0$ is an eigenvector.

<u>Note 2</u>: $\vec{V}_1$ and $\vec{V}_2$ are not co-lineal, as this contradicts the equations in (2).

We now change coordinates, and write vectors in terms of $\vec{V}_1$ and $\vec{V}_2$: $\vec{Y} = u\vec{V}_1 + v\vec{V}_2$, where $\vec{Y} = \begin{pmatrix} x \\ y \end{pmatrix}$. (3)

Note then $A_0 Y = -\omega v \vec{V}_1 + \omega u \vec{V}_2$. (4)

Thus, in terms of the complex variable $z = u + iv$, A_0 becomes multiplication by $i\omega$ ⟵ (5)

Clearly we can write the original system in terms of z, so it takes the form

$$\dot{z} = f_c(u, v, \delta) \qquad (6)$$

for some complex valued, smooth, f_c

We now expand f_1 for z and δ small. Using the fact that we can write u and v in terms of z and $\bar{z}$, the equation takes the form (we also use (5))

$$\dot{z} = i\omega z + \delta(az + b\bar{z}) + (cz^3 + dz^2\bar{z} + ez\bar{z}^2 + f\bar{z}^3) + O(\delta^2, \delta z^2, z^4) \tag{7}$$

where $a, b, \ldots, f$ are (complex) <u>constants</u>, independent of $z, \bar{z}$ and δ.

<u>Simplifying Assumption</u>: Note that there are no quadratic terms in z. This makes the algebra simpler, but is not fundamental. On your own: add the missing terms and carry the algebra.

<u>Expansion</u>: let $\epsilon = \sqrt{|\delta|}$, $0 < \epsilon \ll 1$, and write

$$Z = \epsilon Z_0(t, \mathcal{T}) + \epsilon^3 Z_2(t, \mathcal{T}) + \dots \qquad (8)$$

where $\mathcal{T} = \epsilon^2 t$ and the dependence on t is <u>periodic</u>

<u>Note</u>: if there are quadratic terms you must add $\epsilon^2 Z_1(t, \mathcal{T})$ above!

$O(\epsilon)$ equation $\qquad Z_{0t} = i\omega Z_0, \qquad (9)$

with solution $\qquad Z_0 = A(\mathcal{T}) e^{i\omega t}$

$O(\epsilon^3)$ equation

This should be d!

$$Z_{2t} - i\omega Z_2 = -Z_{0\mathcal{T}} + \sigma a Z_0 + e|Z_0|^2 Z_0 \qquad (10)$$

$$+ \text{Non Resonant Terms (NRT)}$$

where $\boxed{\sigma = \operatorname{sign} \delta}$ and the NRT are terms which depend on t via $e^{-i\omega t}$ and $e^{\pm 3 i \omega t}$ (and yield corresponding dependences in Z_2). The resonant terms, on the hand, have the form This should be d!

$$(-A_{\tau} + \sigma a A + e|A|^2 A)\,e^{i\omega t}$$

and produce terms $\propto$ to $t e^{i\omega t}$ in Z_2. Thus we eliminate them

This should be d!

$$\Rightarrow \quad \underbrace{\frac{d}{d\tau} A = \sigma a A + e|A|^2 A}_{\text{Canonical Form Hopf}} \qquad (11)$$

Write $A = \rho e^{i\varphi}$ (polar) then

$$\frac{d}{d\tau}\rho = \sigma \,\mathrm{Re}(a)\,\rho + \mathrm{Re}(e)\,\rho^3 \qquad (12)$$

These should be d!

$$\frac{d}{d\tau}\varphi = \sigma \,\mathrm{Im}(a) + \mathrm{Im}(e)\,\rho^2 \qquad (13)$$

Because the origin is a spiral for $\delta \neq 0$, we <u>must</u> have $\mathrm{Re}(a) \neq 0$. Also, generically $\mathrm{Re}(e) \neq 0$. This should be d!

Then (12) shows that a Hopf bifurcation occurs as δ crosses 0.

d NOT e

Specifically:

If $\mathrm{Re}(e) < 0$ a supercritical or soft bifurcation occurs, with a stable limit cycle for $\sigma \mathrm{Re}(a) > 0$, corresponding to

$$\rho = \sqrt{-\sigma \mathrm{Re}(a) / \mathrm{Re}(e)}$$

d NOT e

If $\mathrm{Re}(e) > 0$ a subcritical or hard bifurcation occurs, with an unstable limit cycle for $\sigma \mathrm{Re}(a) < 0$ — same formula for ρ.

<u>Note</u> When quadratic terms are included, the same equation in (11) results. However the coefficient of $|A|^2 A$ is not just e, but an expression that involves several coeff. in the expansion.

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