Asymptotic versus convergent

Quite frequently the approximations obtained by perturbation methods are not convergent (thus adding more terms need not improve them) but asymptotic. Here we explore this topic a bit.

Example

Let $\underline{y(x) = e^{1/x}\int_0^x \frac{1}{t} e^{-t^{-1}}\,dt}$ $\quad x>0$ [A]

It is easy to check that this solves

$$x^2\tilde{y}' + \tilde{y} = x \qquad [B]$$

with the restriction "y is bounded for x small and $x>0$". The general sln. is $\tilde{y} = y + c e^{\frac{1}{x}}$. (1)

To prove this note $0 < y < x$ because

$$\int_0^x \frac{1}{t} e^{-\frac{1}{t}}\,dt < \int_0^x \frac{x}{t^2} e^{-\frac{1}{t}}\,dt = x e^{-\frac{1}{x}}$$

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(1) To prove this multiply the equation by $x^{-2}e^{-\frac{1}{x}}$.

Then $\frac{d}{dx} e^{-\frac{1}{x}}\tilde{y} = \frac{1}{x}e^{-\frac{1}{x}}$ QED

Formal series solution to [B]

Substitute $\tilde{y} = \sum_0^\infty a_n x^n$ into [B] and solve for the a_n.

This yields $$\tilde{y} = \sum_0^\infty (-1)^n n!\, x^{n+1} \qquad [C]$$

Exercise: derive this series

The series in [C] does not converge. Yet it yields a very good approximation to $y(x)$ (the solution in [A]) for x small.

Why? The series is asymptotic; it can be shown that

$$y(x) = \sum_0^N (-1)^n n!\, x^{n+1} + O(x^{N+2}) \qquad [D]$$

Thus, as x gets small, for a fixed N, the error vanishes like x^{N+2}

Side note: How do we prove [D]? We use integration by parts, using $e^{-\frac{1}{t}} = t^2 \frac{d}{dt} e^{-\frac{1}{t}}$ in

[A]. This gives

$$y(x) = e^{\frac{1}{x}} \int_0^x t \frac{d}{dt} e^{-\frac{1}{t}} dt = x - e^{\frac{1}{x}} \int_0^x e^{-\frac{1}{t}} dt$$

Then integrate by parts again $\Rightarrow$

$$y(x) = x - x^2 + 2e^{\frac{1}{x}} \int_0^x t e^{-\frac{1}{t}} dt$$

And then again, and so on, to obtain

$$y(x) = \sum_0^N (-1)^n n! \, x^{n+1} + (-1)^{N+1} (N+1)! \, e^{\frac{1}{x}} \int_0^x t^N e^{-\frac{1}{t}} dt$$

Then use

$$0 < \int_0^x t^N e^{-\frac{1}{t}} dt < \int_0^x \frac{x^{N+2}}{t^2} e^{-\frac{1}{t}} dt = x^{N+2} e^{-\frac{1}{x}}$$

multiply integrand by $\frac{x^{N+2}}{t^{N+2}} > 1$

Asymptotic series and expansions are very common. In 1735 Euler used one such expansion to compute the Euler constant γ with 6 decimal digits, and in 1781 he pushed this to 15 digits!

<u>Note</u>: Convergent series provide better approx. as the number of terms grows, and have ∞ accuracy. But they may converge very slowly, so slow as to make them not practical.

Asymptotic series have "finite accuracy", but they may require far less terms and still provide "good enough" accuracy.

<u>DEFINITION</u> $u \sim \sum_0^\infty a_n x^n$ [E_1]

means that the series is asymptotic, so that, for any N, $\sum_0^N a_n x^n = u + O(x^{N+1})$ [E_2]

That is, the error has the size of the first neglected term. But the series need <u>not</u> converge, as in the example above.

<u>Note</u> Asymptotic series need not involve just powers of the small parameter. In

boundary layer theory things like

$$u \sim 1 + a x \log x + b x^2 + \dots \qquad [F]$$

or

$$u \sim 1 + a\sqrt{x} + b x + c\, x/\log x + \dots$$

show up. The point is that each term is smaller than the prior one, and the error behaves like the "next term" in the expansion.

<u>Poincaré's extension</u> In this course we will see expansions of the form

$$u \sim a(t) + \epsilon b(t) + \epsilon^2 c(t) + \dots \qquad [G]$$

where $0 < \epsilon \ll 1$. In this case it does not make sense to require that the error be smaller than the next term, because the next term may vanish (e.g.: $\epsilon^3 \sin t$) for some t. Instead we only require the error to be $O(\epsilon^3)$, or whatever the next term "ϵ behavior" is.