

vdP-trapping

Proof that van der Pol has limit cycle, using trapping regions.

$\epsilon > 0$

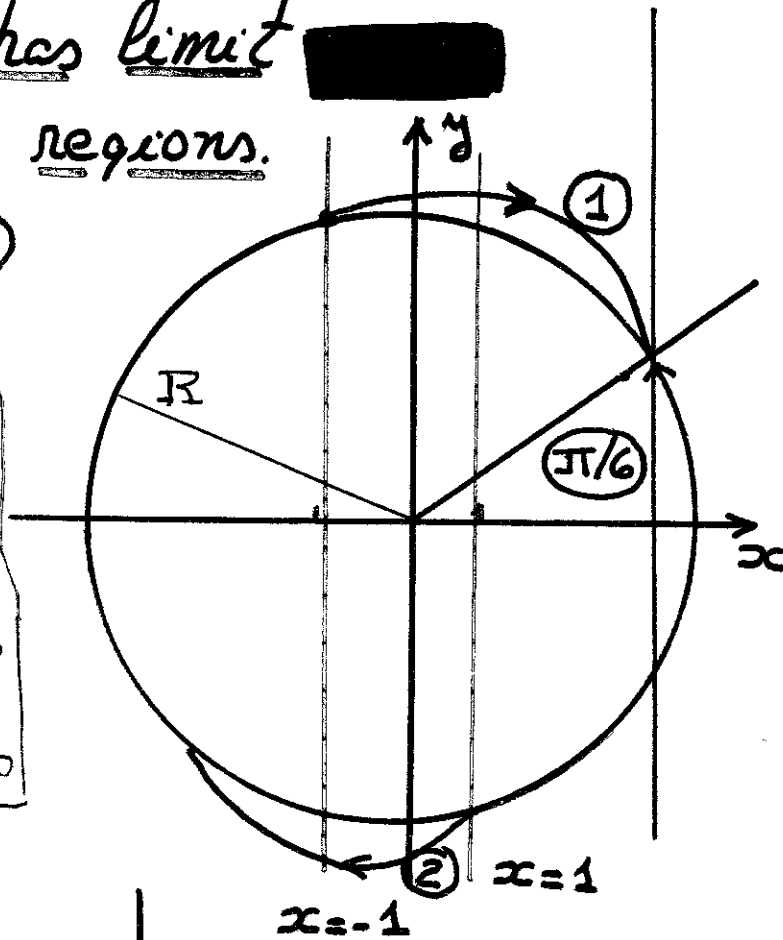
$\dot{x} = y$

$\dot{y} = -x + \epsilon(1-x^2)y$

$r = \sqrt{x^2 + y^2}$

$r \frac{dr}{dt} = \epsilon(1-x^2)y^2$

Note symmetry
 $x \leftrightarrow -x$
 $y \leftrightarrow -y$



$\therefore r \frac{dr}{dx} = \epsilon(1-x^2)y$

Note that for $\frac{1}{6}\pi < \theta < \frac{5}{6}\pi$

$\frac{1}{2}r < y = r \sin \theta \leq r$

For $r > R$ this happens on $-\frac{\sqrt{3}}{2}R < x < \frac{\sqrt{3}}{2}R$

For $|x| > 1$, flow is inward across $r=R$
 want to show orbit 1 & 2 do as shown \Rightarrow trapping region.

$\sin \frac{\pi}{6} = \frac{1}{2}, \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}$

Show now 1 returns to $r \leq R$ before $x = x_0 = \frac{\sqrt{3}}{2}R$, as shown, for R large enough.

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38. 2

$$\text{For } -1 < x < 1: \quad 0 < \cancel{r} \frac{d\Gamma}{dx} \leq \epsilon(1-x^2) \cancel{r}$$

$$\Rightarrow \underline{\underline{\Delta\Gamma_1}} = \int_{-1}^1 \frac{d\Gamma}{dx} dx \leq \underline{\underline{(4/3)\epsilon}}$$

$$\text{For } 1 < x < x_0: \quad \blacksquare \cancel{r} \frac{d\Gamma}{dx} < \frac{\epsilon}{2}(1-x^2) \cancel{r}$$

(as long as $\Gamma \geq R$ and $y > \frac{1}{2}\Gamma$ valid)

$$\therefore \underline{\underline{\Delta\Gamma_2}} = \int_1^{x_0} \frac{d\Gamma}{dx} dx < \frac{\epsilon}{2} \left\{ x_0 - \frac{2}{3} - \frac{1}{3}x_0^3 \right\}$$

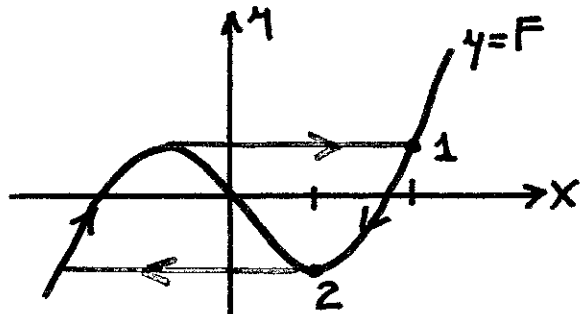
$$\therefore \underline{\underline{\Delta\Gamma_1 + \Delta\Gamma_2}} < \underline{\underline{\frac{\epsilon}{2} \left\{ x_0 - \frac{1}{3}x_0^3 + 2 \right\}}}$$

Now, as $R \rightarrow \infty$, $x_0 \rightarrow \infty$

\therefore for R large enough, $\Delta\Gamma \leq 0$

and orbit must return to $r \leq R$!

Period of van der Pol Relaxation regime



$$\dot{x} = \mu(y - F(x))$$

$$\dot{y} = -\frac{1}{\mu}x$$

$$\mu \gg 1$$

$$T \cong 2 \int_1^2 dt = 2 \int_1^2 \frac{dt}{dx} dx = -2\mu \int_{x_1}^{x_2} \frac{F'(x) dx}{x}$$

$$\left\{ \frac{dt}{dy} \frac{dy}{dx} = -\frac{\mu}{x} \frac{dy}{dx} \right.$$

For van der Pol $F = \frac{1}{3}x^3 - x$, $x_2 = 1$, $x_1 = 2$

$$\Rightarrow T \cong 2\mu \int_1^2 \frac{x^2 - 1}{x} dx = \mu(3 - 2\ln 2)$$