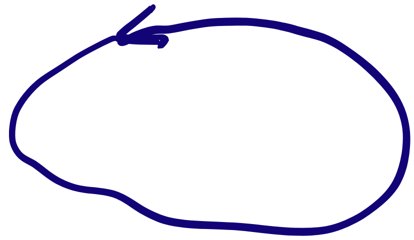
Limit cycle



periodic
isolated orbits

Criteria for showing
no limit cycle — Index Theory ✓
Gradient Systems ✓
Liapunov Funct. ✓
Dulac's Criteria

Yes limit cycle : Trapping regions

Dulac's

$$\dot{x} = f(x,y) \qquad \Big| \quad \frac{d}{dt}\vec{Z} = \vec{V}(\vec{Z})$$
$$\dot{y} = g(x,y)$$

Imagine there is a closed orbit

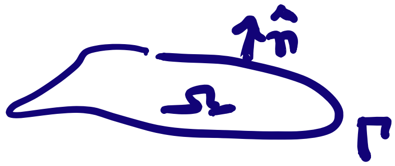
Γ

and $V \parallel \Gamma$

Compute flow of vector field

through Γ

$$\int_\Gamma \vec{v}\cdot\hat{n}\,ds = 0 = \int_\Omega (\operatorname{div}\vec{v})\,dx\,dy$$



$\therefore\ \vec{v}\cdot\hat{n} = 0$ on Γ

$\therefore$ If $\operatorname{div}\vec{v} > 0$ "everywhere"

There cannot be periodic

– orbits –

Same argument applies to

$$\vec{V}_m = g\cdot\vec{V}$$

where g is a scalar function

If you can find a g such that $\operatorname{div}[g\cdot\vec{v}] > 0$

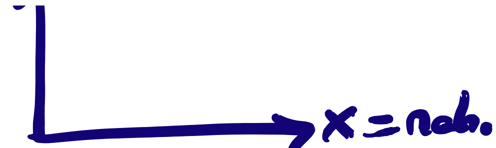
There no periodic sln.

Dulac

EXAMPLES

g = sh.

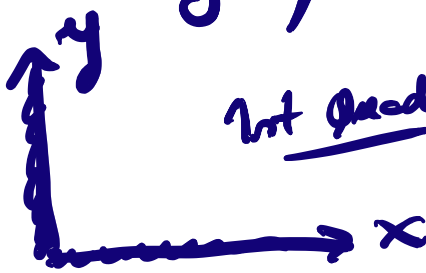
Rabbits / sheep



$$\boxed{\dot{x} = R\left(1-\frac{x}{N}\right)x - I\,xy} \quad + \text{ same for } y$$

$$\dot{x} = ax - bx^2 - dxy = g \qquad b > 0$$

$$\dot{y} = \alpha y - \beta y^2 - \delta xy = f \qquad \beta > 0$$



1st quadrant invariant

$$\boxed{\begin{aligned} \dot{x} &= p\,x \\ \dot{y} &= q\,y \end{aligned}}$$

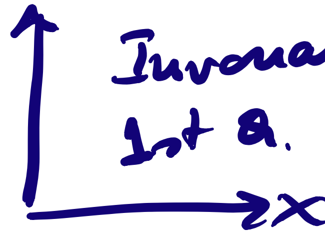
$$\boxed{\frac{g}{xy} = \frac{a}{y} - \frac{bx}{y} - d} \qquad \boxed{\frac{f}{xy} = \frac{\alpha}{x} - \frac{\beta y}{x} - \delta}$$

$$\left(\frac{g}{xy}\right)_x + \left(\frac{f}{xy}\right)_y = -\frac{b}{y} - \frac{\beta}{x} < 0$$

in 1^{st} Q.

$$\dot{x} = (4 - y - x^2)x = f$$

$$\dot{y} = (x-1)y = g$$



Invariant 1st Q.

$$\left(\frac{f}{yx}\right)_x + \left(\frac{g}{xy}\right)_y = \left[\frac{y}{y} - 1 - \frac{x^2}{y}\right]_x + \left[1 - \frac{1}{x}\right]_y$$

$$= -\frac{2x}{y} < 0 \quad \text{on Q1.}$$

$\dot{x} = y - 2x$

$\dot{y} = \mu + x^2 - y$

$(\dot{x})_x + (\dot{y})_y = -3 < 0$

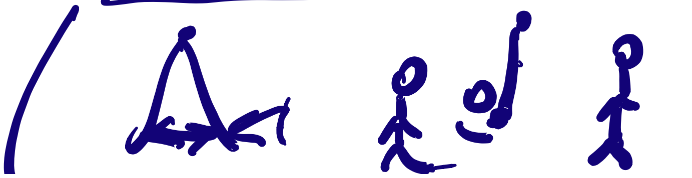
$\boxed{\text{Van der Pol Equ}}$ $\boxed{\ddot{x} + x = 0}$

Add dissp. $\ddot{x} + \nu\dot{x} + x = 0$ ($\nu > 0$ const)

$\boxed{E = \frac{1}{2}\dot{x}^2 + \frac{1}{2}x^2}$ $\frac{dE}{dt} = -\nu\dot{x}^2 \leq 0$

$\ddot{x} + \nu\dot{x} + x = 0$ $\nu = \nu(x)$

$$\boxed{\ddot{x} + \nu(1 - x^2)\dot{x} + x = 0}$$



$\boxed{\text{van der Pol equ}}$

This is a "canonical form" !!

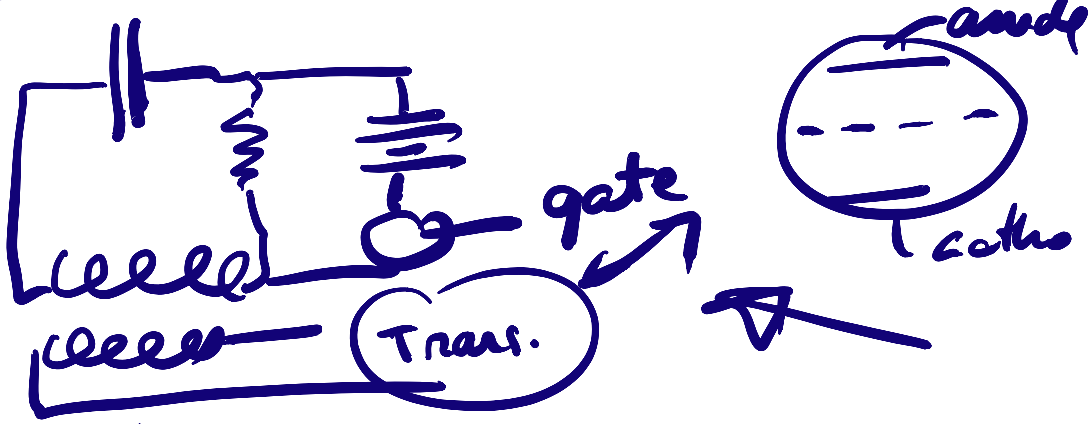
Van der Pol ~ 1926 For electronic circuit

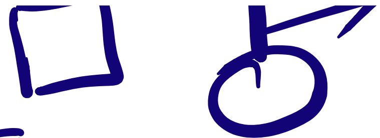
$$\boxed{\dot{x} = r + x^2} \qquad x = \sqrt{r}\,\tilde{x} \quad t = \sqrt{r}\,\tilde{t}$$

$$\boxed{\frac{d\tilde{x}}{d\tilde{t}} = 1 + \tilde{x}^2}$$

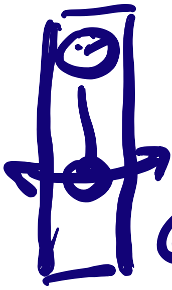
Derivation Van der Pol. See I

Stoker J.J. Nonlinear vibrations in mechanical and electrical systems. Wiley 1992

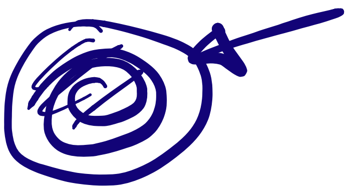


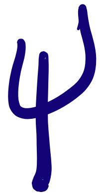


Clocks



Grandfather





$$\boxed{\ddot{x} - \mu(1-x^2)\dot{x} + x = 0}$$ μ const.

If $\mu < 0$ then origin is a stable spiral

Linearize $\ddot{x} - \mu\dot{x} + x = 0$

If $\mu > 0$ then origin is an unstable spiral

Nonlinear term $\mu x^2 \dot{x}$

For $\mu > 0$ this has a unique stable limit cycle

For $\mu < 0$ " " " unique unstable limit cycle

For $\mu \gg 1$ and μ small

limit cycle is almost harmonic

Regime of interest in electronics

Poincare Bendixon Theorem

1. Orbit is critical point
2. " is periodic
3. " approaches C.P. as $t \to \infty$
4. " " periodic orbit as $t \to \infty$
5. " " Cycle graph as $t \to \infty$

This is all that can happen

Also know homeomorphed structurally stable C.P. return type with nonl.

Note it says approaches periodic orbit not limit cycle!

limit cycle $\Longrightarrow$ periodic

$\dot{\theta} = 1$

$\dot{r} = f(r)$