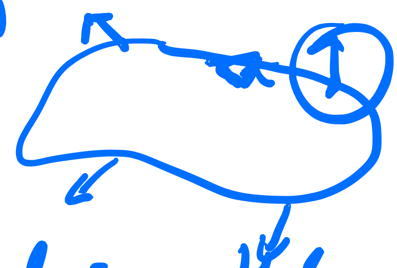
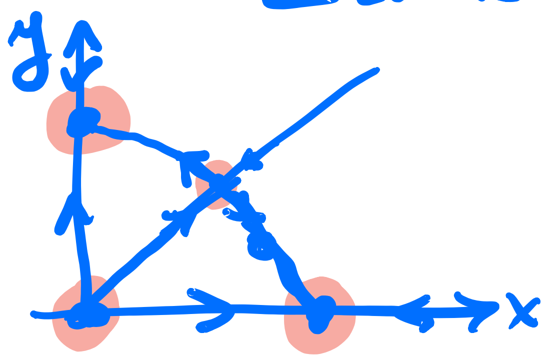
Finish up index theory

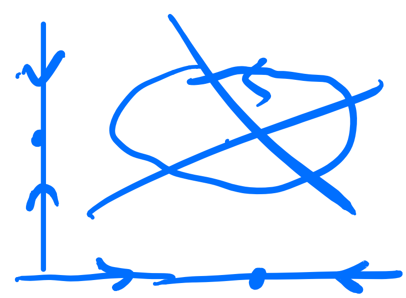
Properties



+ Const if curve moved smoothly without crossing C.P.

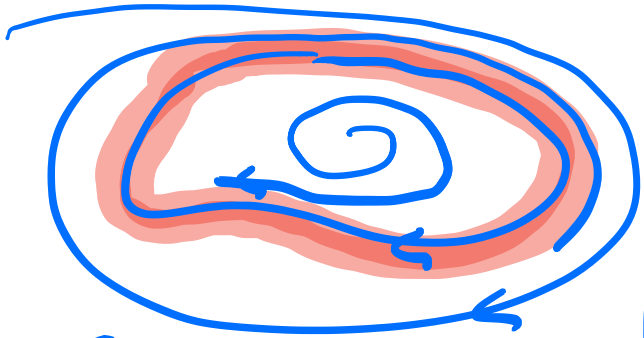
+ Index of a closed orbit is 1
+ Index of a closed curve is equal Σ Index enclosed C.P.





Poincare Bendixson Theorem:
Look at orbit and let $t \to \infty$ on orbit

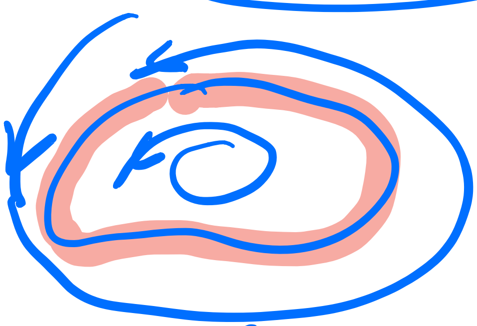
1. It is a C.P.
2. As $t \to \infty$ orbit goes to infinity
3. As $t \to +\infty$ orbit approaches a C.P.
4. Orbit is periodic
5. as $t \to \infty$ orbit approaches a periodic orbit or a cycle graph



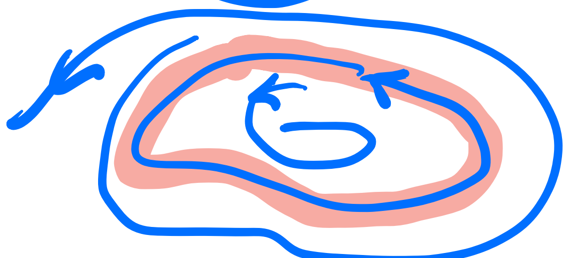
1) periodic orbit

2) isolated

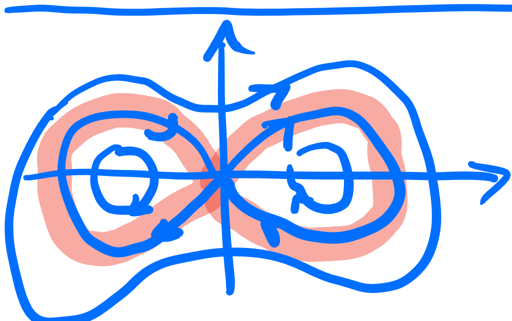
limit cycle



stable



unstable

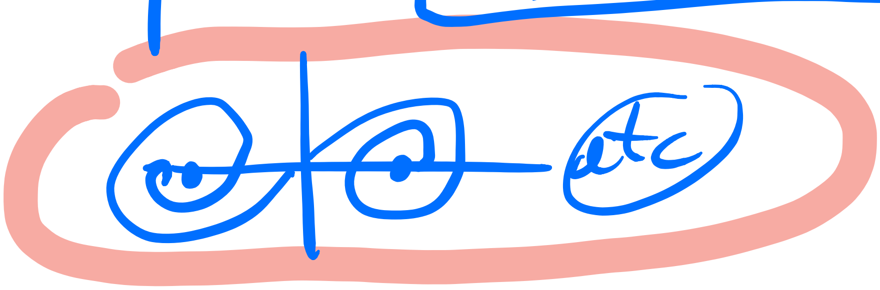


$$\ddot{x} + \frac{\partial V}{\partial x} = 0 \qquad y = \dot{x}$$

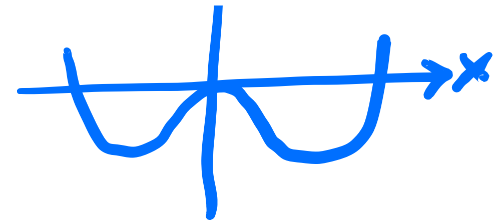
potential $V =$

$$\ddot{x} + \nu \dot{x} + \frac{\partial V}{\partial x} = 0$$

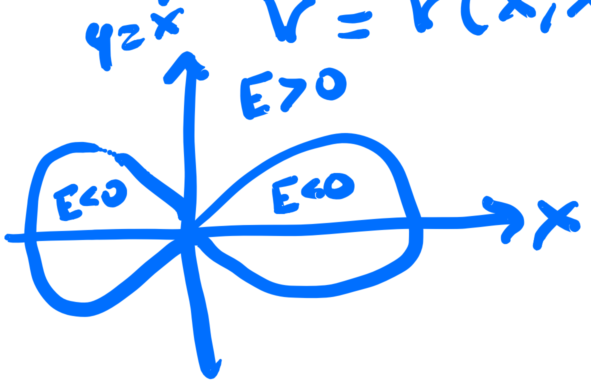
ν = const



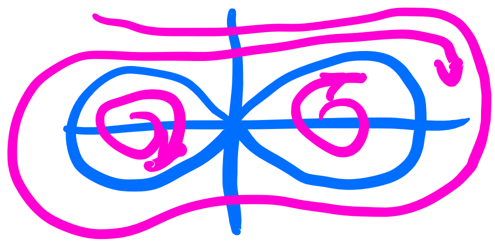
$$\ddot{x} + \nu \dot{x} + \frac{\partial V}{\partial x} = 0$$



$$\nu = \nu(x, \dot{x})$$



$$\boxed{\nu = E}$$



$$\boxed{E = \frac{1}{2}\dot{x}^2 + V(x) = 0}$$

Question A) What is there about a ν that switches sign.

Not the exception

Limit cycles

$$\boxed{\begin{aligned} \dot{x} &= -y \\ \dot{y} &= x \end{aligned}}$$

Harmonic oscillator

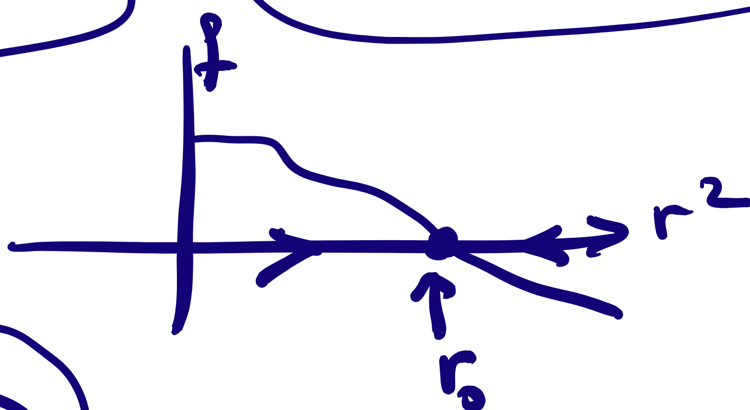
$m = 1$

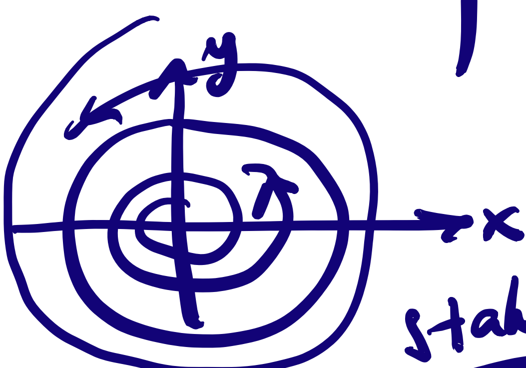
$K = 1$

In polar, this is 1) $\boxed{\begin{aligned}\dot r &= 0\\ \dot\theta &= 1\end{aligned}}$

$$\boxed{\begin{aligned}\dot r &= f(r^2)\\ \dot\theta &= 1\end{aligned}} \qquad \boxed{\begin{aligned}\dot x &= f(r^2)x - y\\ \dot y &= f(r^2)y + x\end{aligned}}$$

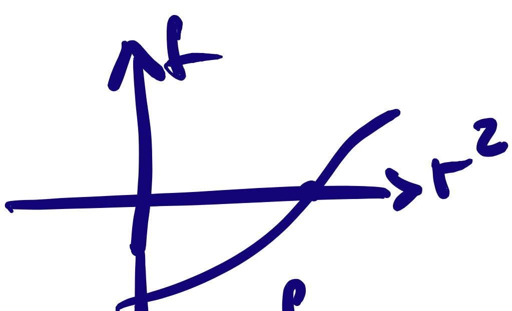
f



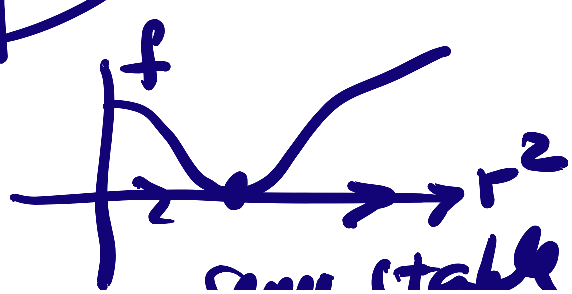


Example of limit cycle

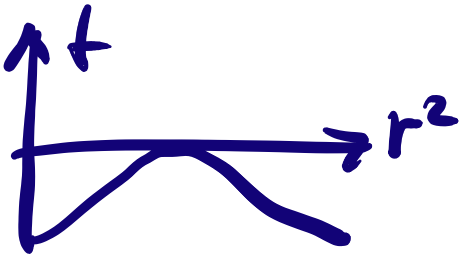
stable

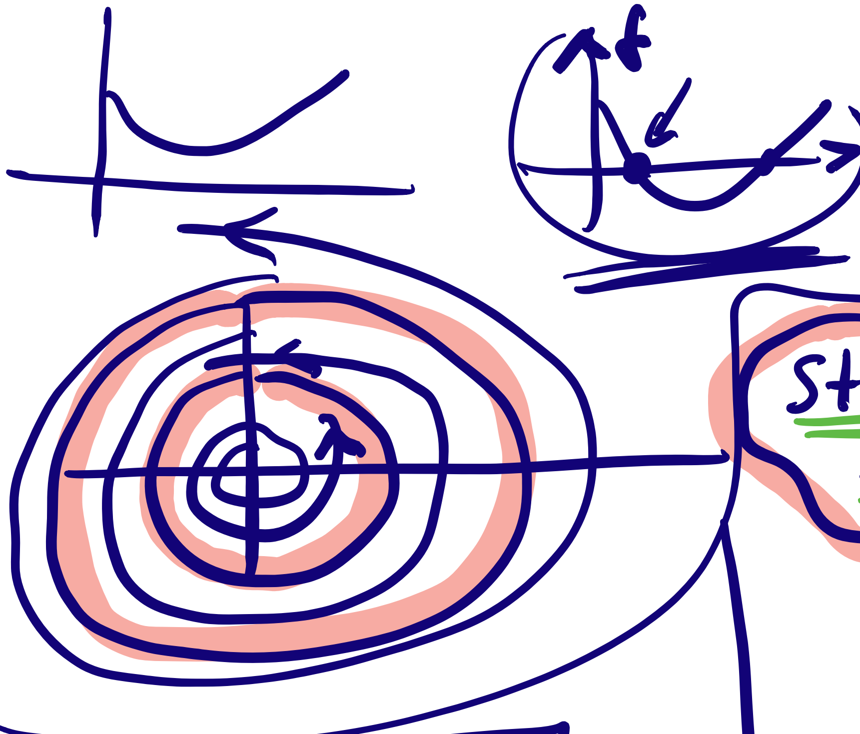


$\Longrightarrow$ unstable l.c.



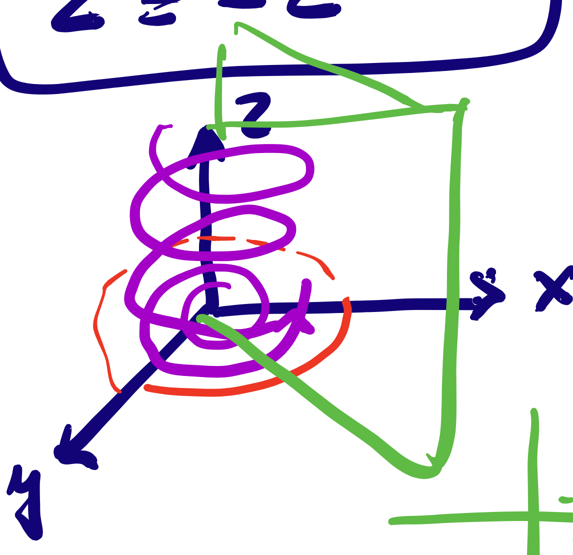
semi stable

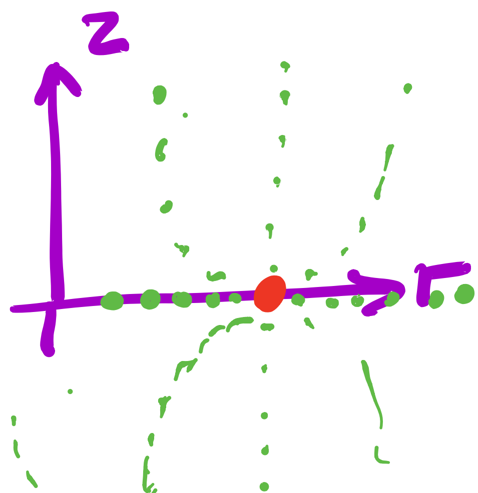


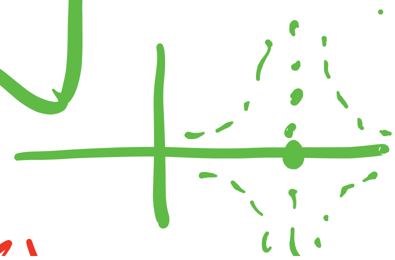


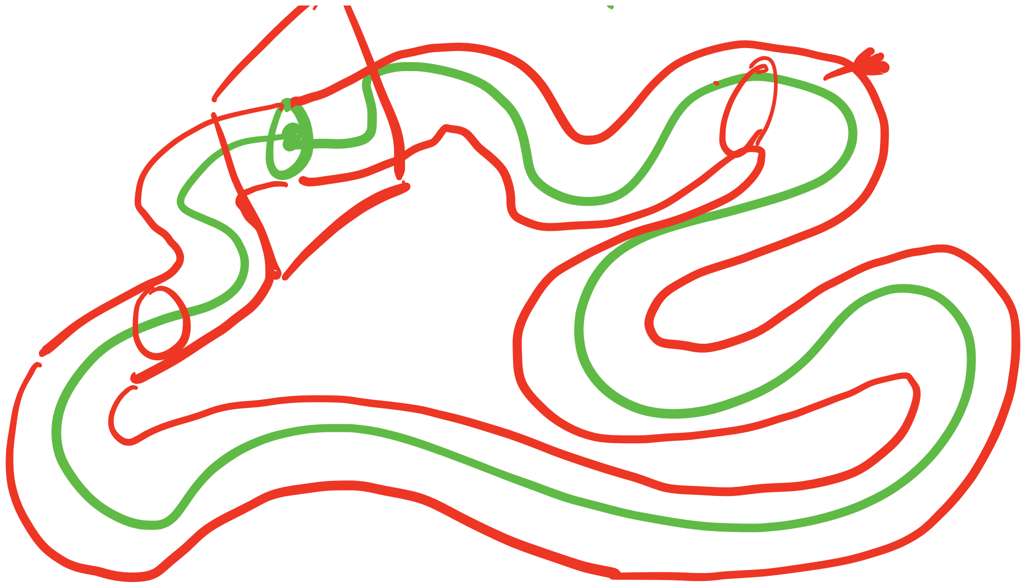
Struct. stability

$$\dot{r} = f(r^2)$$
$$\dot{\theta} = 1$$
$$\dot{z} = \overset{+}{-} z$$

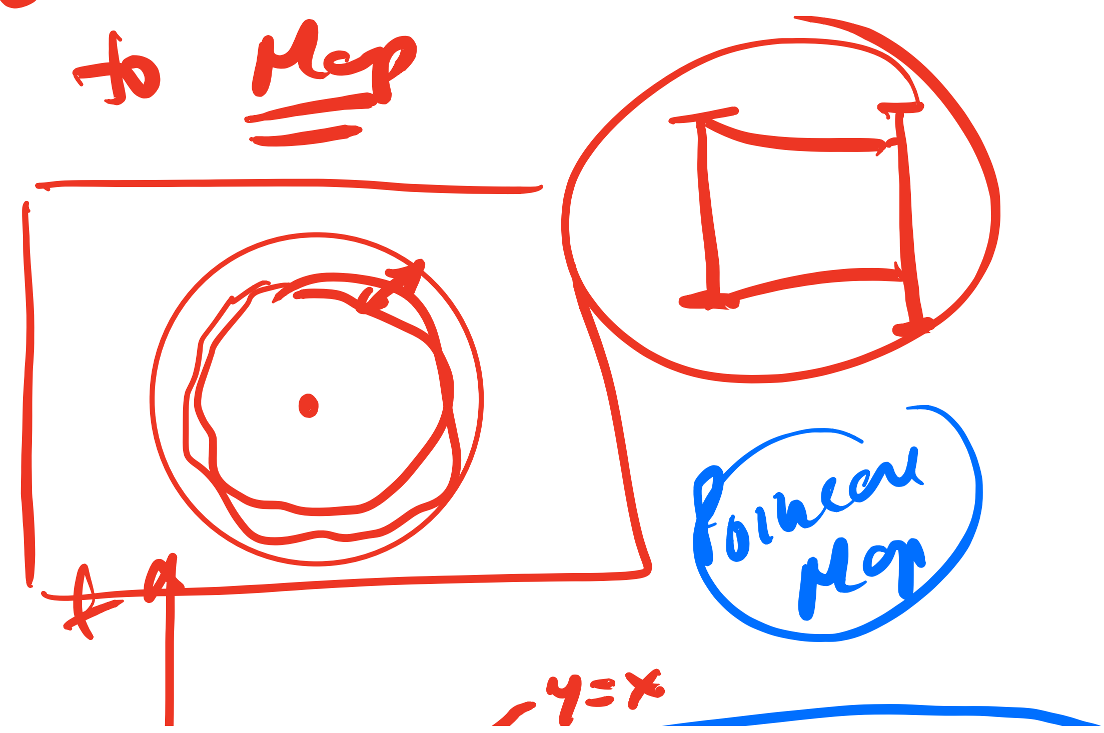






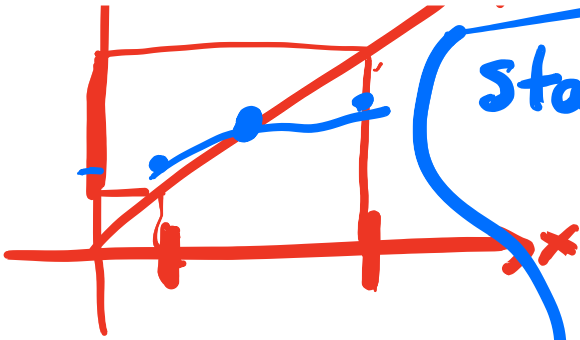


Reduce Cont. Dynamical Syst.
to **Map**



Poincare Map

$y = x$



Stable limit cycle is also structurally stable

Have surface S — x_0

Take point in S

Solve $\dot{\vec{x}} = \vec{F}(\vec{x})$ starting at x_0

then find where orbit intersects S again. This $f(x_0)$

$\vec{x} = \vec{\sigma}(t)$ periodic solution of system

$\vec{\sigma}(t+T) = \vec{\sigma}$

$\dot{x} = F(\vec{x})$

Take solution where $\vec{x}(0) = \vec{\sigma}(0) + \delta \vec{x}_0$

$x = \sigma + \delta x$

$\cancel{\dot{\sigma}} + \dot{\delta x} = \cancel{F(\sigma)} + J(\sigma)\,\delta x$

$$\boxed{\delta \dot{x} = J(t)\,\delta x}$$ Floquet theory

Criteria for limit cycles (Existence)

1. Index Theory

2. Systems that do not have limit cycles

1. Gradients systems $V = V(\underset{\sim}{x})$

$$\boxed{\dot{x} = -\nabla V}$$

$$\frac{d}{dt} V(x) = (\nabla V) \cdot \frac{dx}{dt} = -(\nabla V)^2$$

2. If you have a lyapunov function $L = L(\underset{\sim}{x})$

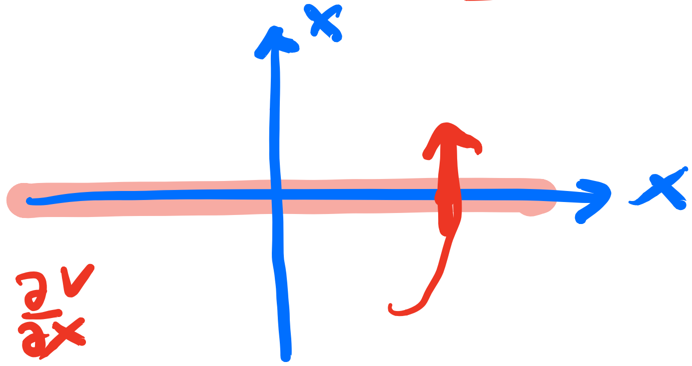
$\frac{dL}{dt} < 0$ on away from C.P.

$$\boxed{\frac{dL}{dt} \le 0}$$

$$\boxed{\ddot{x} + \nu \dot{x} + \frac{\partial V}{\partial x} = 0}$$ $\nu > 0$ const.

$$L = \frac{1}{2}\dot{x}^2 + V(x)$$

$$\frac{dL}{dt} = \ddot{x}\dot{x} + V'(x)\dot{x} = -\nu \dot{x}^2 \le 0$$



$\dot{x} = y$

$\dot{y} = -\nu y - \frac{\partial V}{\partial x}$

$\frac{dL}{dt} \le 0$ and $\frac{dL}{dt} = 0$ only on isolated point on orbit

→ orbit $\dot{x} = F$, $F \neq 0$ where

$\cancel{\frac{dL}{dt} = 0}$

$\frac{dL}{dt} = 0$