Structural Stab. bifurcations

$\boxed{\text{Recap}}$

Saddle node

Normal form $\dot{x} = r + x^2$

perturb $\dot{x} = h + r + x^2$

$\rightarrow$ S.S. ✓

Transcritical $\dot{x} = rx - x^2$

Arb. pert. $\dot{x} = h + rx - x^2$

No S.S.

Restrict so $x=0$ remains solution $\dot{x} = (r+h)x - x^2$

Now S.S. ✓

Pitchfork $\dot{x} = rx \pm x^3$

Generic pert.: $\dot{x} = h + rx \pm x^3$

Destroys pitchfork

Restrict so $x=0$ remains soln | $\dot{x} = rx + hx^2 \pm x^3$

No pitchfork

Example: rod under pressure but rod not symmetric

Restrict to keep symmetry

$$\boxed{\dot{x} = (r+h)x \pm x^3}$$

s.s.v

Next topic bifurcations involving a C.P. in d dims.

$\dot{Y} = F(Y, r)$ | $F(Y_0, r_0) = 0$

Y & F vectors

We now look at Y & r close to Y_0, r_0

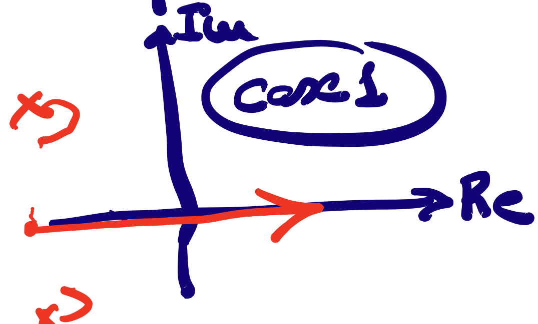
Stability Only situations that are stable make sense. This means that: On one side of r_0 we must have a stable C.P. i.e. all the e.v. of $\frac{\partial F}{\partial Y}$ have negative real parts.

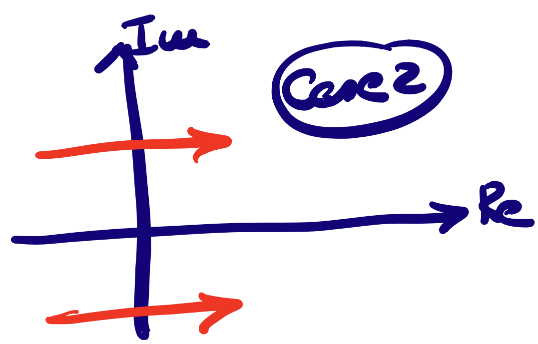
At bifurcation we expect a loss of stability $\therefore$ Generic situation

① At $r=r_0,\ Y=Y_0$ $\dfrac{\partial F}{\partial Y}$ has a simple zero eigenvalue with a single eigenvector

② At $r=r_0,\ Y=Y_0$ $\dfrac{\partial F}{\partial Y}$ has a pair of pure imaginary eigenvalues

In pictures





Arrows indicate motion of ev. as r changes

We do now interaction (1)

Generically $\longrightarrow$ Saddle Node

Saturation (2) later

Generically $\longrightarrow$ Hopf bif.

See pdf with Calculation in web page to supplement lecture below

First we compute the zeros of F, $F(Y, r) = 0$ for Y close to Y_0 and r close to r_0

Next we derive the normal form

$\dot{Y} = F(Y, r)$ $\quad F(Y_0, r_0) = 0$

$A = F_Y(Y_0, r_0)$ sing.

and A has only ONE zero eig. with multiplicity one

$$\boxed{\begin{array}{l} AR = 0 \quad R \neq 0 \\ \uparrow \text{right ev.} \\ LA = 0, \\ L \cdot R = 1 \end{array}}$$

One more generic assump

$F_r(Y_0, r_0) = b$ $\quad L \cdot b \neq 0$

All this $\Rightarrow$ Saddle node

$$F(Y, r) = 0$$

r close to r_0

Y close to Y_0

$$F(Y_0 + dY, r_0 + dr) = \underline{A}\, dY + b\, dr + Q(Y, Y) + \text{H.O.T.}$$

$$\|Y\| = O(\epsilon)$$

$$dr = O(\epsilon^2) \qquad 0 < \epsilon \ll 1$$

$$Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$$

$$\boxed{r = r_0 + \epsilon^2 \sigma}$$

$\sigma = \pm 1 \qquad \epsilon = \sqrt{|r - r_0|}$

$$0 = \epsilon A Y_1 + \epsilon^2 A Y_2 + \epsilon^2 \sigma b + \epsilon^2 Q(Y_1, Y_1) + \underline{O(\epsilon^3)}$$

$\underline{O(\epsilon)}$ $\boxed{A Y_1 = 0}$ $Y_1 = aR$

some $\underline{a}$

const.

$O(\epsilon^2)$

$$\underline{0 = A Y_2 + \sigma b + a^2 Q(R, R)}$$

$$\underline{A Y_2 = -\sigma b - a^2 Q(R, R)}$$

when does this have a $\underline{\text{soln}}$?

$\underline{\text{Fredholm alternative}}$

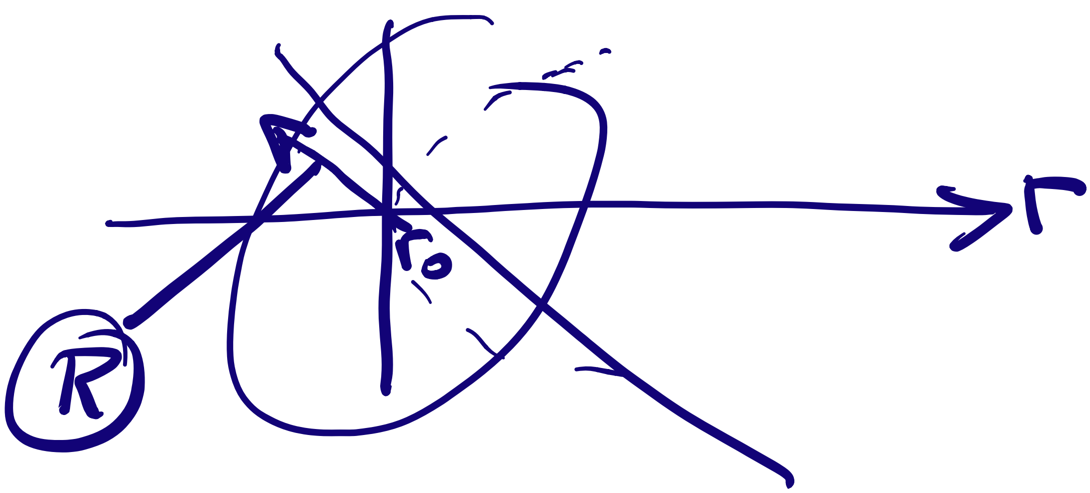
$L A Y_2 = 0$ means $\underline{LA = 0}$

$$0 = \underbrace{\sigma(L.b)}_{\text{non zero scalar}} + a^2 \underbrace{L.\Phi(R,R)}_{\text{queue}}$$

$$\boxed{a^2 = \frac{L.b}{L.\Phi(R,R)}\sigma}$$

sign( ) $= \sigma$

$$\epsilon^3 A Y_3 = \text{w. b. of c.}$$



$$\boxed{\dot{a} = \delta r + a^2}$$

$$Y = Y_0 + \epsilon Y_1(T) + \epsilon^2 Y_2(\ )$$

$$\boxed{r = r_0 + \sigma\epsilon^2} \quad \underline{\underline{0 < \epsilon << 1}}$$

$$T = \epsilon t$$

Y_2 = exp. decay on time scale t + slow T component

$$\boxed{Y_2(T, t)}$$

$$\underline{\underline{\frac{dY}{dt} = \epsilon^2 \underline{Y_{1T}} + \epsilon^2 Y_{2t} + O(\epsilon^3)}}$$

$O(\epsilon)$ $\quad AY_1 = 0 \quad Y_1 = a(T)R$

$O(\epsilon^2)$ $\quad \dot{a}R$

$$Y_{2t} = AY_2 + \sigma b + a^2 Q(R, R)$$

$$Y_2 = \sum \alpha_n e^{\lambda_n t} R_n + \tilde{Y}_2(T)$$

$$\dot{a}R = A\tilde{Y}_2 + \sigma b + a^2 Q(R, R)$$

$$\dot{a} = \sigma L b + (L Q(R, R)) a^2$$

Normal form Saddle Node