<u>Bifurcation</u> <u>calculation</u> $\dot{Y} = F(Y, r)$

$$\underbrace{F(Y_0, r_0) = 0 \ \& \ A = F_Y(Y_0, r_0) \text{ singular}}_{\text{bif. condition}}$$

$\boxed{\text{Generic case}} \Rightarrow$

① Multiplicity one singularity $\quad AR = 0$, $\quad LA = 0$

Normalize $L \cdot R = 1$

$L$ = left eigenvector
$R$ = right eigenvector

② $b = F_r(Y_0, r_0) \neq 0$ and $L \cdot b \neq 0$

$\boxed{\text{Stability}}$ all the other e.v. of $A$ satisfy $\mathrm{Re}(\lambda) < 0$.

Now

$$F(Y_0 + dY, r_0 + dr) = A\,dY + b\,dr + \underbrace{Q(Y, Y)}_{\text{quadratic form}} + O(Y\,dr, dr^2)$$

We need to go to quadratic in $Y$ because there is a direction where $A\,dY = 0$.!

Dominant balance now yields $d\Gamma = O(\|Y\|^2)$

$\therefore$ expand solution to $F(Y,\Gamma)=0$ as follows

$$Y = Y_0 + \epsilon Y_1 + \epsilon^2 Y_2 + \cdots$$

$$\Gamma = \Gamma_0 + \epsilon^2 \sigma + \cdots$$

where $\sigma = \pm 1$. Then

$$F = \epsilon A Y_1 + \epsilon^2 A Y_2 + \epsilon^2 \sigma b + \epsilon^2 Q(Y_1, Y_1) + O(\epsilon^3)$$

Thus

$O(\epsilon)$ $AY_1 = 0$ i.e.: $Y_1 = aR$,

$a$ some constant

$O(\epsilon^2)$ $AY_2 + \sigma b + Q(R,R)a^2 = 0$

Left multiply by $L$ to get condition for solution : $\underline{\sigma(L\cdot b) + a^2 L\cdot Q(R,R) = 0}$

Generic condition #3 $L \cdot Q(R,R) \neq 0$

Then $a^2 = -\sigma \dfrac{L \cdot b}{L \cdot Q(R,R)}$

determines both $a$ and the sign of $\sigma$.
(Note two solutions)

Dynamics in this case we take

$$Y_1 = Y_1(T) \qquad T = \epsilon t$$

$$Y_2 = \tilde{Y}_2(T) + \text{exponential decaying component}$$

This yields $Y_1 = a(T)\,R$
with $a$ satisfying the normal form for saddle nodes.

Note $\epsilon t$ is the "critical slowing down" time scale.