

Bifurcations

- Saddle node: no assumptions
- Transcrit.: a soln. always exists "steady state"

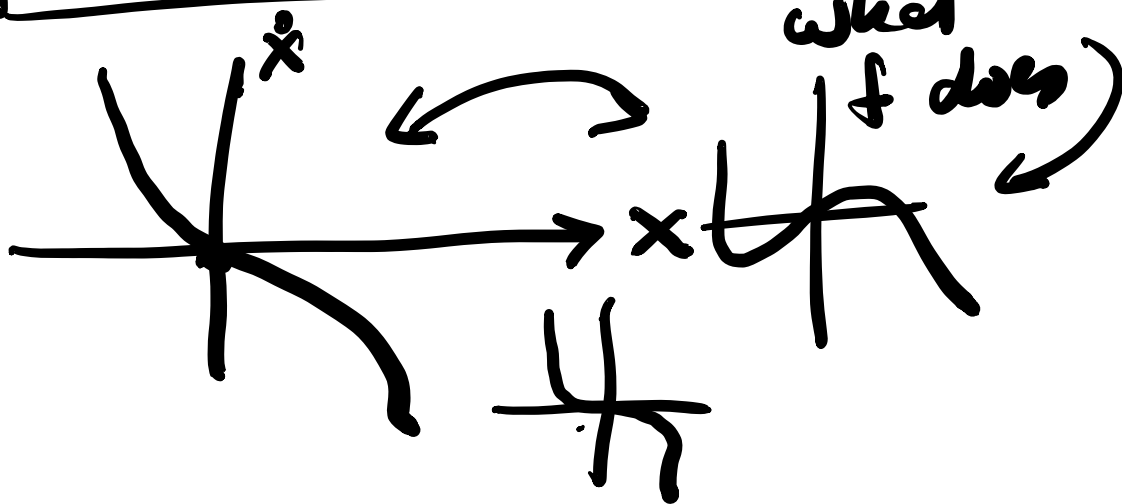
Now
assume

$$\dot{x} = f(x, r)$$



$$f(-x, r) = f(x, r)$$

$$\dot{x} = rx + x^3 \quad | \quad \text{Pitchfork}$$



Normal form

$$\dot{x} = rx \pm x^3$$

+ → Nonl. is destabilizing

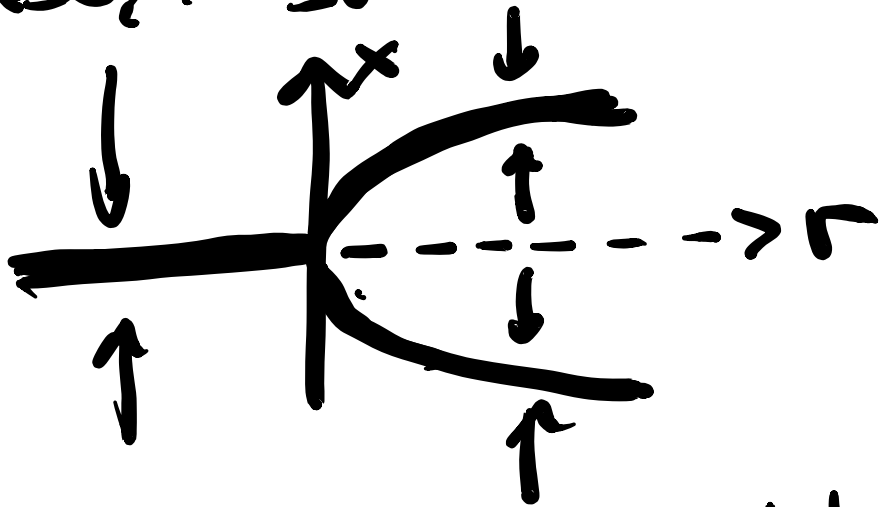
- → Nonl. is stabilizing

$\dot{x} = rx - x^3$

Stabilizing N.L.
Soft Pitchfork

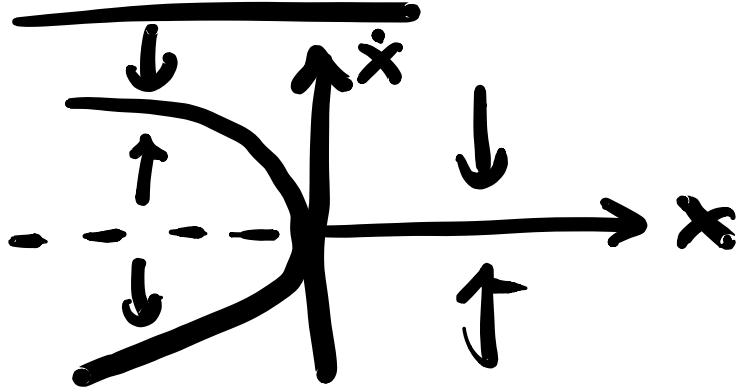
$rx - x^3 = 0$

$x = 0, x = \pm\sqrt{r}$ when $r > 0$

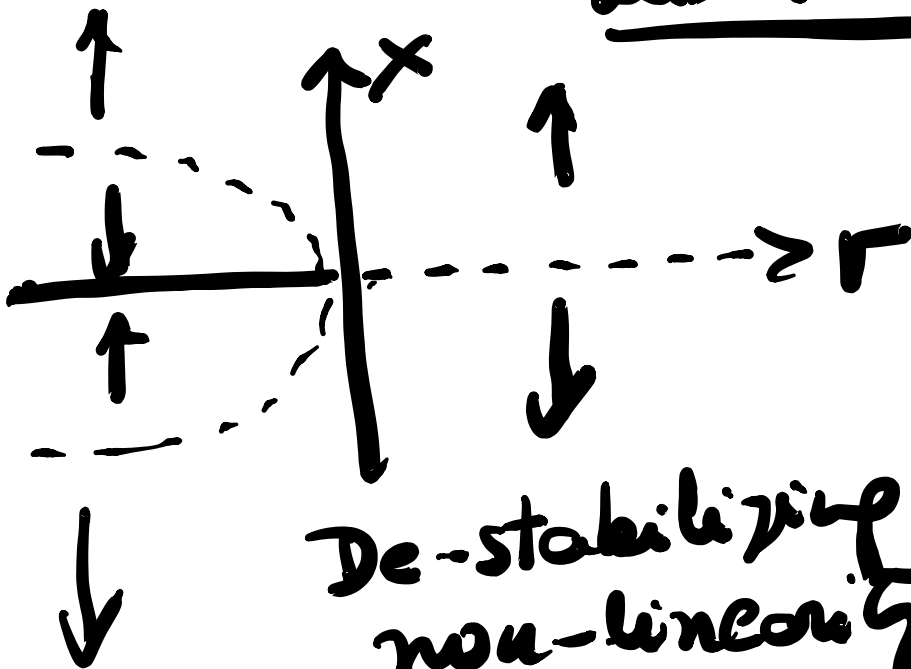


Supercritical pitchfork
Stabilizing nonlinearity

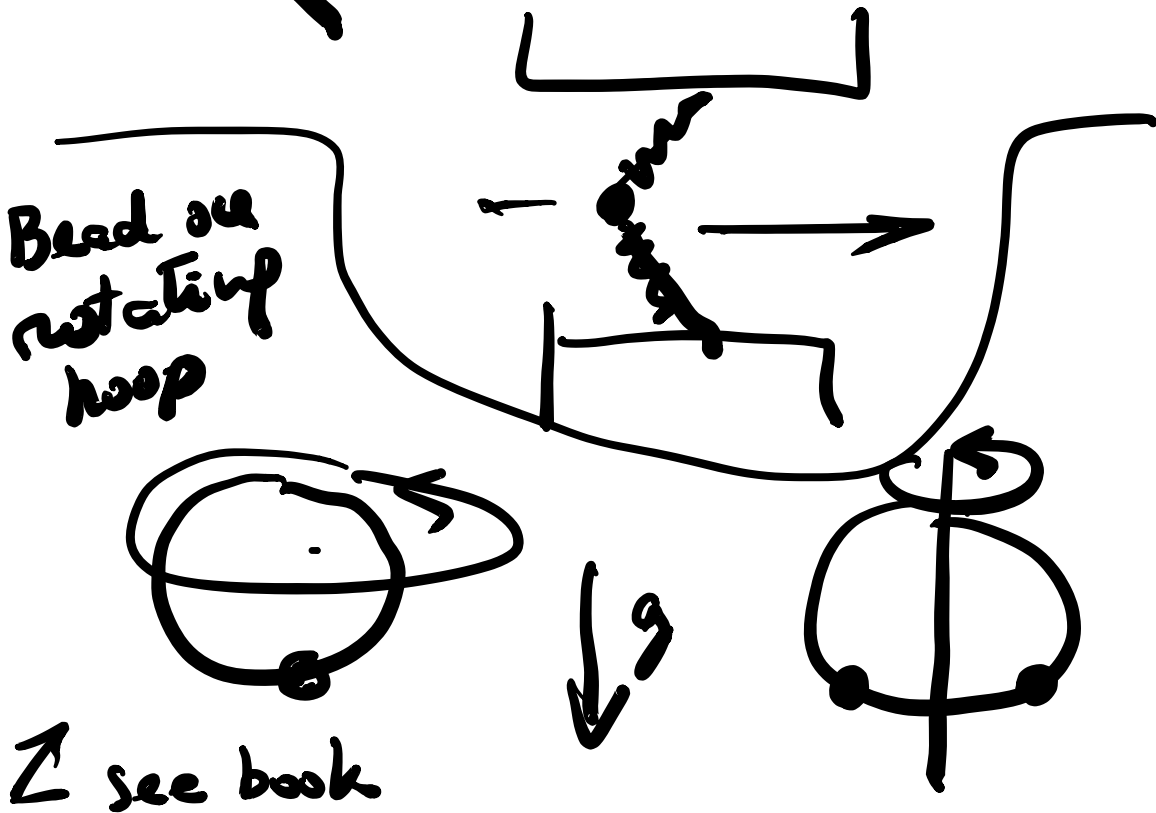
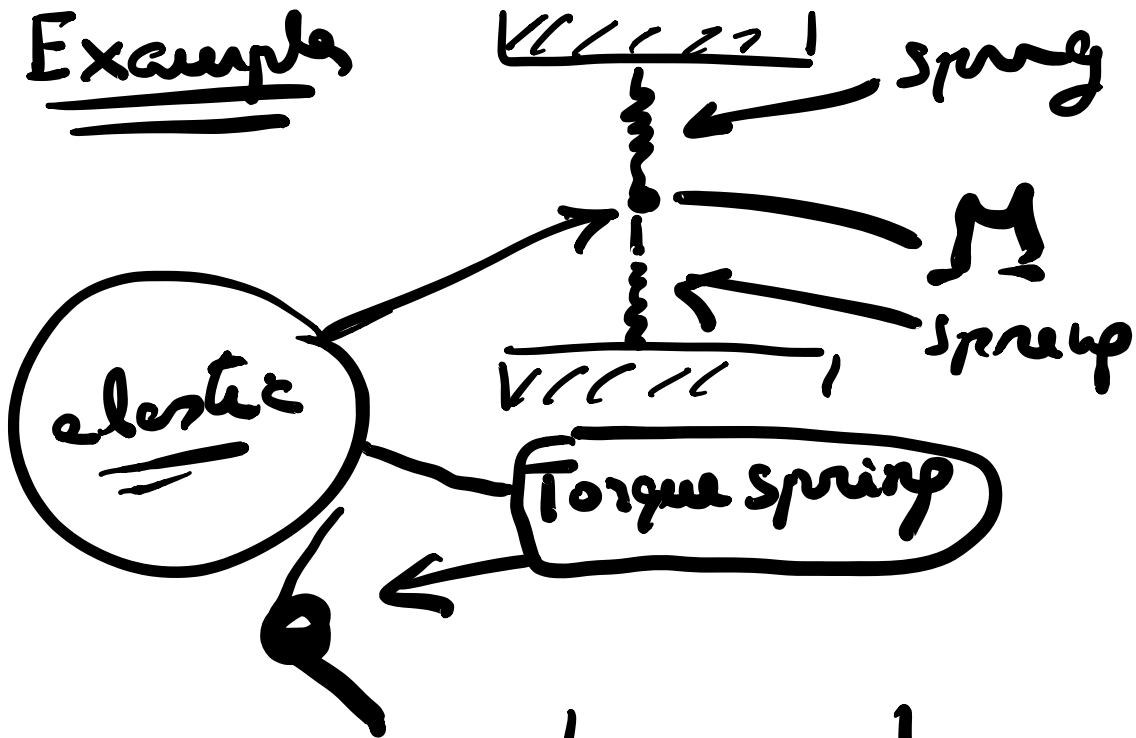
$$\dot{x} = -\Gamma x - x^3 \quad \underline{\text{Soft}} \text{ Pitchfork}$$



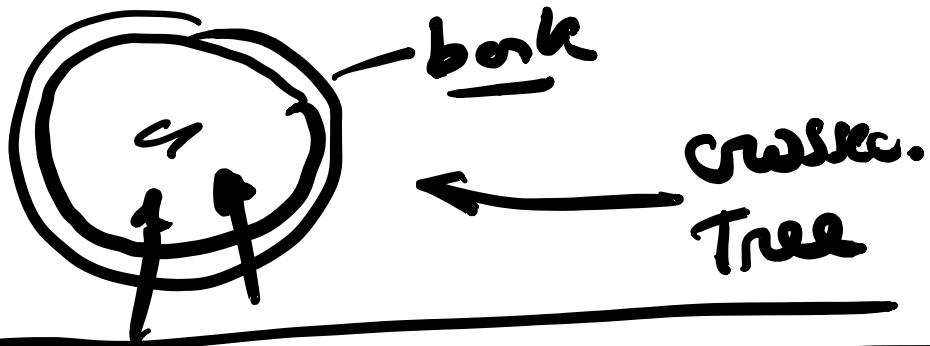
$$\dot{x} = \Gamma x + x^3 \quad \text{hard Pitchfork} \\ \text{or } \underline{\text{Subcritical}}$$



Example



Spring Worm example:
see book



Structural Stability

$$\dot{x} = f(x, r)$$

$$f(x, r) = 0$$

$$\Rightarrow \underline{x = x(r)}$$

$$f(0, 0) = 0$$

$$f_x(0, 0) = 0$$

$$\begin{matrix} |x| \ll 1 \\ |r| \ll 1 \end{matrix}$$

Justify normal form

$$f = \underbrace{f(0, 0)}_{=0} + \underbrace{f_r}_0 + \underbrace{f_x}_0 x + \frac{1}{2} \underbrace{f_{xx}}_0 x^2 + \dots$$

$$\dot{x} = a\tau + bx^2 \quad \begin{matrix} a \neq 0 \\ b \neq 0 \end{matrix}$$

↳ Normalize, $\dot{x} = \tau + x^2$

Add $f(0, \tau) \equiv 0$

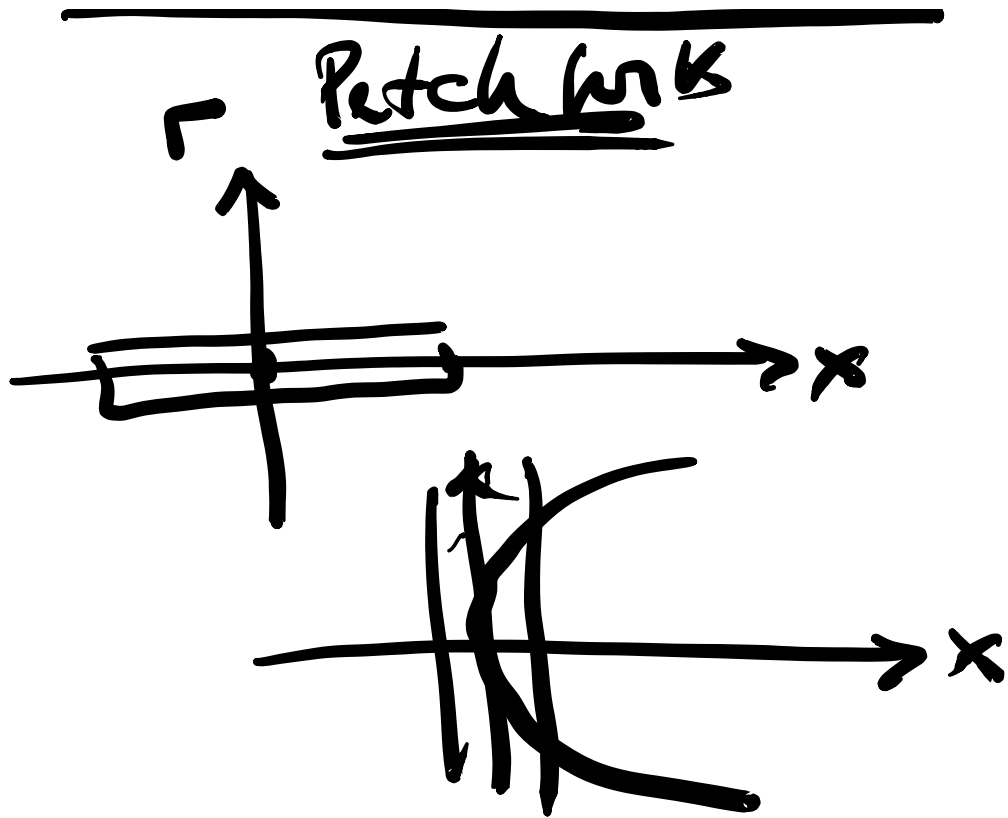
$$f(x, \tau) = \frac{1}{2} f_{\tau x}^0 \tau x + \frac{1}{2} f_{xx}^0 x^2$$

* Trans. Non. Form

Add $f(x, \tau) = -f(-x, \tau)$

$\dot{x} = f(x, \tau)$

$$f \sim \frac{1}{2} f_{\tau x}^0 \tau x + \frac{1}{6} f_{xxx}^0 x^3$$

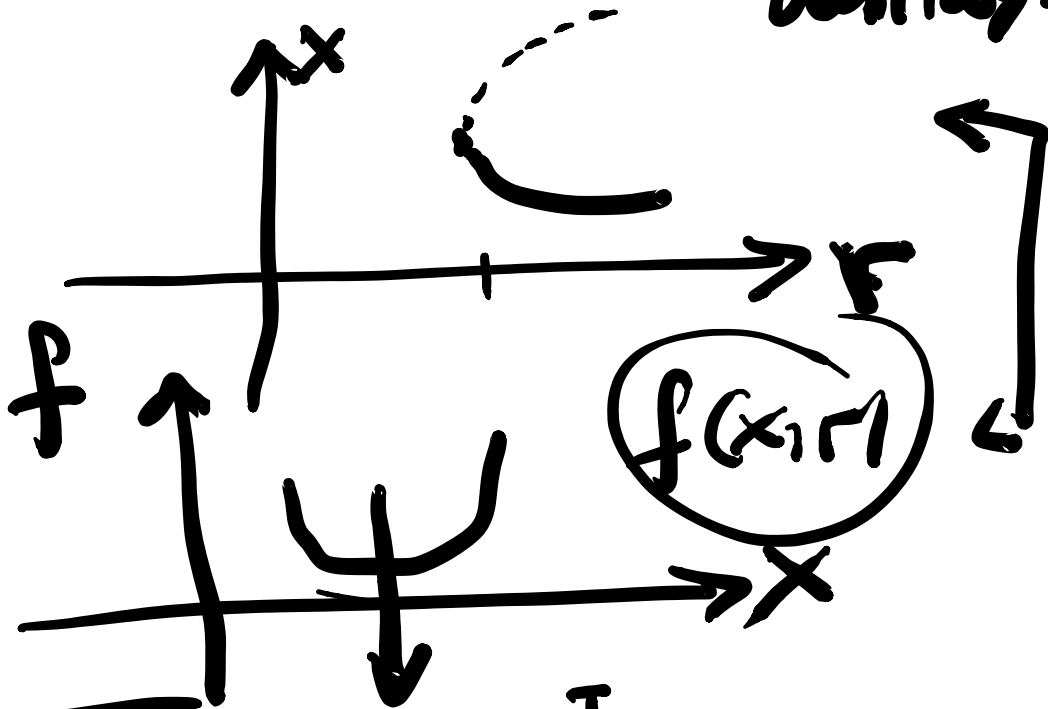


Struct. Stats [3 arguments]

- ① Hand waving ✓
 - ② Looked zero in complex plane ✓
 - ③ Normal forms ✓
-

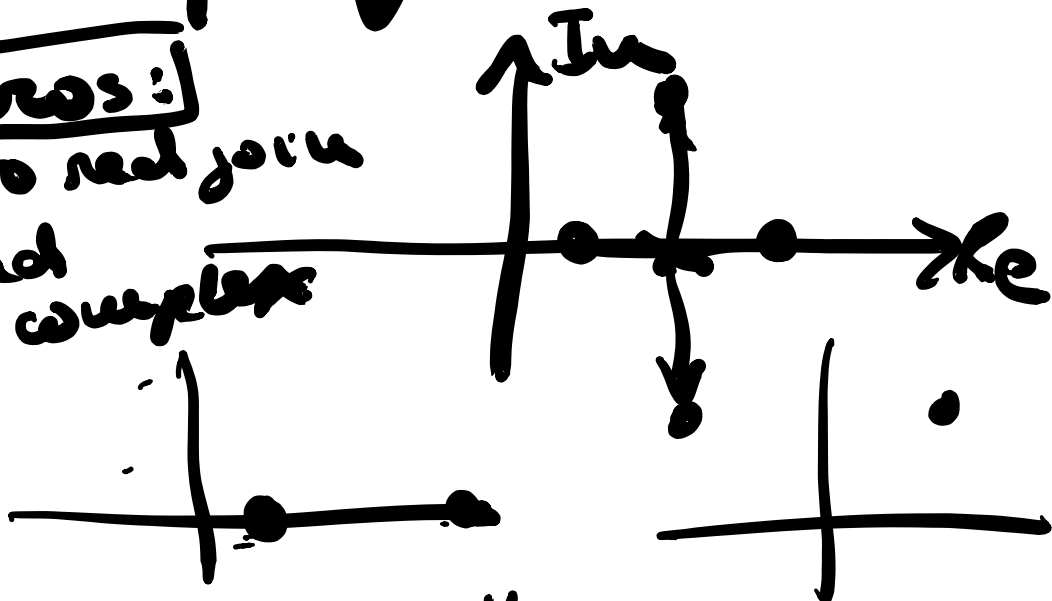
Saddle node: (Hand waving)

Move curve around cannot destroy it



Zeros:

Two real join
and
go complex



Agars: no smooth way
to change this

$$\dot{x} = f(x, r, h) \quad f(0,0,0) = 0$$

$$f_x(0,0,0) = 0$$

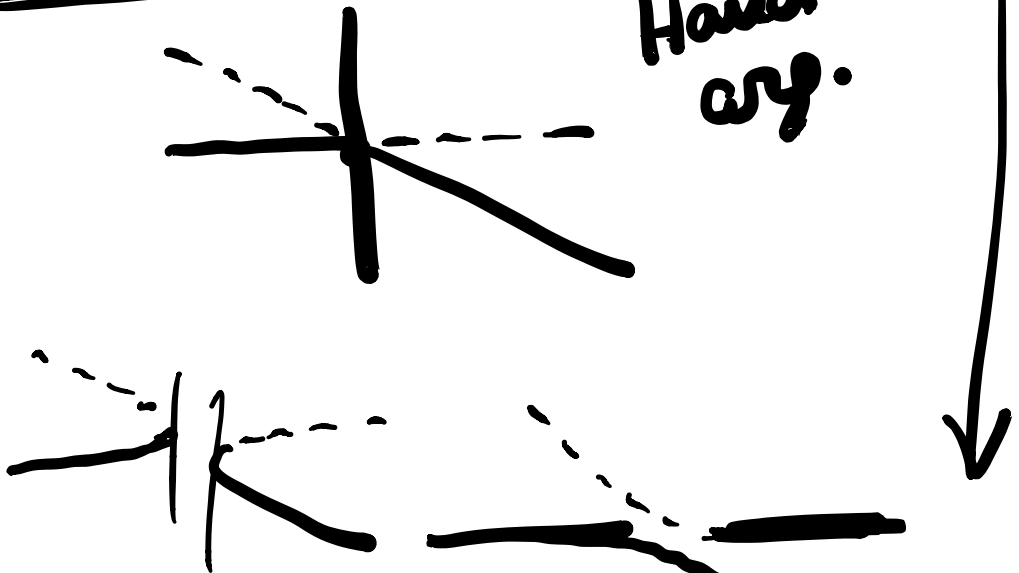
$$f = \left(\frac{f}{h} + \frac{f}{r} \right) + \frac{1}{2} f_{xx} x^2$$

Normal form argu.

$$f = \alpha + x^2$$

only change
is what
is α !

Transcritical



Can change picture
if allowed to not have a
zero fixed. Not S.S.

But cannot change picture
otherwise

Continue next lecture