Answers: 18.376 Problem Set #02, MIT Spring 2023

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1.1 Statement: Edge waves for a pice-wise constant wave equation

Consider the wave equation

$$\boldsymbol{u_{tt} - c^2\,(u_{xx} + u_{yy})) = 0}, \quad \text{for } \boldsymbol{y > 0}, \tag{1.1}$$

where $\boldsymbol{c = c_2}$ for $\boldsymbol{y > L}$, $\boldsymbol{c = c_1}$ for $\boldsymbol{0 < y < L}$, $\boldsymbol{c_2 > c_1 > 0}$ are constants, and $\boldsymbol{L > 0}$ is a constant. Furthermore, **u and u_y are continuous across the $y = L$ interface, and the boundary condition $u = 0$ applies at $y = 0$.**

Find all the edge waves for this problem. Namely, non-vanishing solutions to the problem above of the form

$$\boldsymbol{u = U(y)\ \exp\left(i\,(k\,x - \omega\,t)\right)}, \tag{1.2}$$

where $\boldsymbol{k > 0}$ and $\boldsymbol{\omega}$ are real constants, and $\boldsymbol{U}$ vanishes (exponentially) as $\boldsymbol{y \to \infty}$.

Hint. For any given fixed k, the problem will lead to an eigenvalue problem, with eigenvalue ω^2, and eigenfunction $U = U(y)$. This problem can be then reduced to a transcendental equation that ω^2 must satisfy. In order to study the solutions to this later equation, it may be useful to write it down in terms of the variable $\Delta = (L/c_1)\sqrt{\omega^2 - k^2\,c_1^2}$, which is restricted to the range

$$0 < \Delta < \Delta_M = k\,(L/c_1)\sqrt{c_2^2 - c_1^2}, \tag{1.3}$$

because it must be that $\boldsymbol{k^2\,c_1^2 < \omega^2 < k^2\,c_2^2}$. By the way: **you MUST show that this restriction on ω^2 is needed.** You cannot use this just because I said so here.

1.2 Answer: Edge waves for a pice-wise constant wave equation

Substituting (1.2) into (1.1) leads to the eigenvalue problem (for any given $\boldsymbol{k}$), with eigenvalue $\boldsymbol{\omega^2}$,

$$\boldsymbol{-c^2\,U'' + (k^2\,c^2 - \omega^2)\,U = 0}, \tag{1.4}$$

where $\boldsymbol{U(0) = 0}$, $\boldsymbol{U}$ is not identically zero, $\boldsymbol{U}$ and $\boldsymbol{U'}$ are continuous across $\boldsymbol{y = L}$, and $\boldsymbol{U}$ vanishes as $\boldsymbol{y \to \infty}$.

Let us first show that the eigenvalue problem above can have a solution only if

$$k^2\,c_1^2 < \omega^2 < k^2\,c_2^2. \tag{1.5}$$

- **a.** Case $\boldsymbol{0 \le \omega^2 \le k^2\,c_1^2}$. Then $k^2\,c^2 - \omega^2 \ge 0$ everywhere. Hence: (i) If $U'(0) > 0$, $U'(y)$ is a non-decreasing function, and the condition that U vanishes as $y \to \infty$ cannot be satisfied. (ii) If $U'(0) < 0$, $U'(y)$ is a non-increasing function, and the condition that U vanishes as $y \to \infty$ cannot be satisfied. (iii) If $U'(0) = 0$, $U \equiv 0$.
- **b.** Case $\boldsymbol{k^2\,c_2^2 < \omega^2}$. Then U is a sinusoidal for $y \ge L$. It can vanish as $y \to \infty$ only if it vanishes identically. But then U has to vanish everywhere.
- **c.** Case $\boldsymbol{k^2\,c_2^2 = \omega^2}$. Then U is a linear function for $y \ge L$. It can vanish as $y \to \infty$ only if it vanishes identically. But then U has to vanish everywhere.

Given (1.5), the fact that $U(0) = 0$, and the fact that $U \to 0$ as $y \to \infty$, we can write

$$U = \alpha \sin\left(\frac{\sqrt{\omega^2 - k^2\, c_1^2}}{c_1}\, y\right) \quad \text{for } \; 0 \le y \le L, \tag{1.6}$$

$$U = \beta \exp\left(-\frac{\sqrt{k^2\, c_2^2 - \omega^2}}{c_2}\, (y - L)\right) \quad \text{for } \; L \le y, \tag{1.7}$$

where α and β are two constants, both of which cannot vanish. Continuity of U at $y = L$ gives

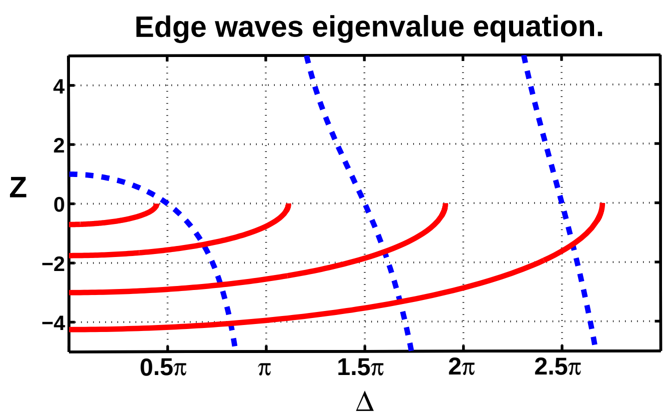
$$\beta = \alpha \, \sin \Delta, \quad \text{where} \quad \Delta = \frac{\sqrt{\omega^2 - k^2\, c_1^2}}{c_1}\, L. \tag{1.8}$$

Continuity of U' at $y = L$ gives

$$\Delta\, \cos \Delta = -\frac{\sqrt{k^2\, c_2^2 - \omega^2}}{c_2}\, L\, \sin \Delta = -\frac{c_1}{c_2}\, \sqrt{\Delta_M^2 - \Delta^2}\, \sin \Delta, \tag{1.9}$$

where [1]

$$0 < \Delta < \Delta_M = \frac{k\, L}{c_1}\, \sqrt{c_2^2 - c_1^2}. \tag{1.10}$$



For several values of Δ_M, with $c_2 = 2\, c_1$.
Dashed (blue) lines: plots of the left hand side in (1.11) .... $\boldsymbol{Z = \Delta\, \cos \Delta / \sin \Delta}$.
Solid (red) lines: plots of the right hand side in (1.11) .... $\boldsymbol{Z = -(c_1/c_2)\, \sqrt{\Delta_M^2 - \Delta^2}}$.

Figure 1.1: Edge wave eigenvalue equation.

Writing (1.9) in the form

$$\Delta\, \frac{\cos \Delta}{\sin \Delta} = -\frac{c_1}{c_2}\, \sqrt{\Delta_M^2 - \Delta^2}, \tag{1.11}$$

it is easy to see that (see figure 1.1)

1. For $\boldsymbol{\Delta_M \le \frac{\pi}{2}}$ .... **there is no solution for $\boldsymbol{\Delta}$ in the range given by (1.10),**
2. For $\boldsymbol{\frac{\pi}{2} < \Delta_M \le \frac{5\,\pi}{2}}$ .... **there is one solution for $\boldsymbol{\Delta}$ in the range given by (1.10),**
3. For $\boldsymbol{\frac{5\,\pi}{2} < \Delta_M \le \frac{9\,\pi}{2}}$ .... **there are two solutions for $\boldsymbol{\Delta}$ in the range given by (1.10),**

and so on.

2 Elastic Hanging String

2.1 Statement: Elastic hanging string

Consider an elastic string, with **constant mass per unit length $\boldsymbol{\rho}$**, and **constant cross-sectional area $\boldsymbol{A}$**. Assume small deformations, so that the elastic forces (tension) generated on any small portion of the string have the form

$$T = E\, A\, \frac{\Delta \ell}{\ell} = \kappa\, \frac{\Delta \ell}{\ell} \tag{2.1}$$

[1] Note that $0 < \Delta < \Delta_M$ is equivalent to (1.5).

where ℓ is the un-stretched length of the string segment, $\Delta\ell$ is the length change, E is the Young's modulus for the string material, and $\boldsymbol{\kappa = E\,A}$. Notice that we ignore any changes in A that may occur because of the stretching.

Assume now that the string is hanging vertically, and straight — no lateral displacements, from some fixed point, so that **it can be described by a function $\boldsymbol{Z = Z(\zeta,\, t)}$,** where

A. $\boldsymbol{z}$ is the vertical coordinate, $\boldsymbol{z = 0}$ is the position of the point to which the string is attached, and the relaxed length of the string is $\boldsymbol{L}$

B. $\boldsymbol{z = \zeta}$, $-L \leq \zeta \leq 0$, would be the vertical position of a mass element along the string when not stretched. Thus ζ serves as a label for the mass elements of the string.

C. $\boldsymbol{z = Z(\zeta,\, t)}$ is the actual position of the mass element whose label is ζ. Hence the displacement field along the string is given by $\boldsymbol{u = u(\zeta,\, t) = Z - \zeta}$.

Perform the following tasks:

1. Derive an equation for $\boldsymbol{u}$, assuming that the only external force on the string is gravity, characterized by $\boldsymbol{g}$.
2. What are the boundary conditions for the equation derived in **1**?
3. How do the boundary conditions change if there is a mass $\boldsymbol{m}$ attached to the lower end of the string?
4. What is the equilibrium (no motion) state for the string, as described by the equation and boundary conditions in **2** and **3**? Call this solution $\boldsymbol{u_* = u_*(\zeta)}$.
5. The fundamental modes of vibration for the system in this problem are described by solutions of the form $\boldsymbol{u = u_*(\zeta) + a\ \cos(\omega\, t)\ \sin(k\,\zeta)}$, where (obviously) $a,\, k \neq 0$. **Find equations for $\boldsymbol{\omega}$ and $\boldsymbol{k}$.** *Why is the ζ dependence via a sine?*

2.2 Answer: Elastic hanging string

1. From equation (2.1), the tension along the string is given by

$$T = \kappa\,\frac{(dZ - d\zeta)}{d\zeta} = \kappa\, u_\zeta. \tag{2.2}$$

Consider now an arbitrary section of the string, described by $a < \zeta < b$. The conservation of momentum on this section can be written in the form

$$\frac{d}{dt}\int_a^b \rho\, u_t\, d\zeta = T(b,\, t) - T(a,\, t) - g\,\rho\,(b-a) = \int_a^b (T_z - g\,\rho)\; d\zeta. \tag{2.3}$$

Since this must be true for any string segment, it follows that

$$\boldsymbol{\rho\, u_{tt} - \kappa\, u_{\zeta\,\zeta} = -g\,\rho,} \quad \text{or} \quad \boldsymbol{u_{tt} - c^2\, u_{\zeta\,\zeta} = -g,} \tag{2.4}$$

where $\boldsymbol{c = \sqrt{\kappa/\rho}}$ is the wave speed, and we have used (2.2) to express T_ζ.

2. At the upper, attached, end of the string there should be no motion. At the lower, free, end of the string the tension should vanish (since there is no force that can balance T there). Hence, **the boundary conditions for equation (2.4) are**

$$\mathbf{u(0,\, t) = 0} \quad \text{and} \quad \mathbf{u_\zeta(-L, t) = 0}. \tag{2.5}$$

3. **If there is a mass m attached to the lower end of the string,** then the tension there must be such that it balances the forces produced by that mass: weight and inertia. Hence, in this case **the boundary conditions for equation (2.4) are**

$$\mathbf{u(0,\, t) = 0} \quad \text{and} \quad \mathbf{m\, u_{tt}(-L, t) = -m\, g} + \kappa\, \mathbf{u_\zeta(-L, t)}. \tag{2.6}$$

Obviously, this reduces to (2.5) for $m = 0$.

4. The equilibrium state for the string, as described by the equation and boundary conditions above, is obtained by setting all the time derivatives to zero, and solving the resulting o.d.e. for u. The answer is

$$\boldsymbol{u} = \left(\frac{\boldsymbol{1}}{\boldsymbol{2}}\gamma\left(\frac{\zeta}{\boldsymbol{L}}\right)^{\boldsymbol{2}} + \gamma\left(\boldsymbol{1}+\frac{\boldsymbol{m}}{\boldsymbol{M}}\right)\left(\frac{\zeta}{\boldsymbol{L}}\right)\right)\boldsymbol{L}, \tag{2.7}$$

where $\boldsymbol{M = \rho\, L}$ is the total mass of the string, and $\boldsymbol{\gamma = \dfrac{g\, M}{\kappa}}$ is a non-dimensional parameter measuring the relative strengths of the two forces in the problem.

5. We now consider the fundamental modes of vibration. Substituting into the equation and boundary condition, it is easy to see that it should be $\boldsymbol{\omega = k\, c}$, with

$$\frac{m}{M}(k\, L) = \frac{\cos(k\, L)}{\sin(k\, L)}. \tag{2.8}$$

The case $m = 0$ is trivial, and can be solved explicitly.
Notes:
— *The dependence of the mode on ζ is via a sine because it must be $u = 0$ at $\zeta = 0$.*
— *Strictly speaking, the time dependence for the mode should be written as $\cos\omega\,(t - t_0)$.*
— *The equation relating ω and k, $\omega^2 = c^2\, k^2$, has two roots. However, they give the same answer, so we selected $\omega = k\, c$.*

3 Gravity water waves (dispersion relation)

3.1 Statement: Gravity water waves (dispersion relation)

The equations for (infinitesimal) irrotational surface waves on a liquid over a flat impermeable bottom, when surface tension and dissipative effects are neglected, are

$$\begin{aligned}
\Delta\Phi &= 0, \quad \text{for } 0 < z < h \quad \text{(incompressibility)}. &(3.1)\\
\Phi_z &= 0, \quad \text{for } z = 0 \quad \text{(impermeable bottom)}. &(3.2)\\
\eta_t - \Phi_z &= 0, \quad \text{for } z = h \quad \text{(kinematic boundary condition)}. &(3.3)\\
\Phi_t + g\,\eta &= 0, \quad \text{for } z = h \quad \text{(dynamic boundary condition)}. &(3.4)
\end{aligned}$$

Here (i) $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator, (ii) $\Phi = \Phi(x,\, y,\, z,\, t)$ is the velocity potential — the flow velocity is given by $\vec{u} = \operatorname{grad}\Phi$, (iii) $\vec{x} = (x,\, y)$ are the horizontal coordinates, (iv) z is the vertical coordinate — $z = 0$ is the bottom and $z = h$ is the equilibrium level for the liquid (h is a constant), (v) η is the deviation from equilibrium of the surface — the surface is at $z = h + \eta(x,\, y,\, t)$, and (vi) g is the acceleration of gravity. Equation (3.2) is the statement that there is no flow through the bottom, equation (3.3) states that the velocity of the surface normal to itself is equal to the flow velocity normal to the surface, and equation (3.4) follows from the balance of forces at the interface — Bernoulli's principle.

Compute the dispersion relation for these equations: separate the time and horizontal dependence as $\boldsymbol{e^{i\,\theta}}$ — where $\boldsymbol{\theta = \vec{k}\cdot\vec{x} - \omega\, t}$ and $\boldsymbol{\vec{x} = (x,\, y)}$, solve for the vertical dependence, and find the equation that relates $\boldsymbol{\omega}$ and $\boldsymbol{\vec{k}}$.

3.2 Answer: Gravity water waves (dispersion relation)

Write $\boldsymbol{\Phi = \phi(z)\, e^{i\,\theta}}$ and $\boldsymbol{\eta = a\, e^{i\,\theta}}$, where $\boldsymbol{a}$ is a constant (complex wave amplitude). The equations then become

$$\begin{aligned}
\phi'' - k^2\,\phi &= 0, \quad \text{for } 0 < z < h, &(3.5)\\
\phi' &= 0, \quad \text{for } z = 0, &(3.6)\\
i\,\omega\, a + \phi' &= 0, \quad \text{for } z = h, &(3.7)\\
i\,\omega\,\phi - g\, a &= 0, \quad \text{for } z = h, &(3.8)
\end{aligned}$$

where $k^2 = k_1^2 + k_2^2 = \|\vec{k}\|^2$. Solving (3.5) and (3.6) we obtain $\boldsymbol{\phi = A\,\cosh(k\,z)}$, where A is a constant. Substituting this into (3.7) and (3.8) then yields

$$i\,\omega\,A\,\cosh(k\,h) = g\,a \quad \text{and} \quad \boldsymbol{\omega^2 = g\,k\,\tanh(k\,h)}. \tag{3.9}$$

4 Narrow band wave packages #01

4.1 Statement: Narrow band wave packages #01

Consider a solution to a 1-D linear dispersive system with a narrow band spectrum and a single branch of the dispersion relation active.[2] Using the Fourier Transform, this means that the solution can be written in terms of a scalar function of the form

$$u = u(x,\,t) = \int_{-\infty}^{\infty} U(k)\,e^{i\,(k\,x - \omega\,t)}\,dk, \qquad \text{where:} \tag{4.1}$$

a. The wave-frequency $\omega = \omega(k)$ is given by the dispersion relation. Assume that ω is a smooth, real valued, function.

b. The complex amplitude U is concentrated near some wave-number k_0.

That is[3]
$$U = \frac{1}{\epsilon}\,A\left(\frac{k - k_0}{\epsilon}\right), \tag{4.2}$$
where $0 < \epsilon \ll 1$ and A is a smooth function that decays rapidly at infinity.
We use a-dimensional variables, otherwise a statement like $0 < \epsilon \ll 1$ *has no meaning.*

Assume that $\boldsymbol{\dfrac{d^2\omega}{dk^2}(k_0) = 0}$ and $\boldsymbol{\mu_0 = \dfrac{d^3\omega}{dk^3}(k_0) \neq 0}$. Thus the Taylor expansion for ω centered at k_0 has the form

$$\omega(k) = \omega_0 + c_0\,(k - k_0) + \frac{1}{6}\,\mu_0\,(k - k_0)^3 + \ldots, \tag{4.3}$$

where $\omega_0 = \omega(k_0)$ and $c_0 = c_g(k_0) = \dfrac{d\omega}{dk}(k_0)$ is the group speed at k_0. Then **show that** u in (4.1) has the form of a modulated carrier wave

$$\boldsymbol{u = a(X,\,T)\,e^{i\,(k_0\,x - \omega_0\,t)}}, \tag{4.4}$$

where $\boldsymbol{X = \epsilon\,x}$, $\boldsymbol{T = \epsilon\,t}$, and the modulation amplitude satisfies the equation

$$\boldsymbol{a_T + c_0\,a_X - \frac{1}{6}\,\epsilon^2\,\mu_0\,a_{XXX} = O(\epsilon^3)}. \tag{4.5}$$

What equation does $\boldsymbol{a}$ satisfy in terms of the variables $\boldsymbol{\chi = X - c_0 T}$ and $\boldsymbol{\tau = \epsilon^2\,T = \epsilon^3\,t}$?

Hint. Write $k = k_0 + \epsilon\,\kappa$ *and substitute this into (4.2–4.3). Then use the result in (4.1).*

4.2 Answer: Narrow band wave packages #01

Substituting $k = k_0 + \epsilon\,\kappa$ into (4.3) yields

$$\omega(k) = \omega_0 + \epsilon\,c_0\,\kappa + \epsilon^3\,\frac{1}{6}\,\mu_0\,\kappa^3 + \ldots, \tag{4.6}$$

Then, using this and (4.2) in (4.1) yields

$$u = \underbrace{\int_{-\infty}^{\infty} A(\kappa)\,e^{i\,(\kappa\,X - \Omega\,T)}\,d\kappa}_{a}\;e^{i\,(k_0\,x - \omega_0\,t)}, \tag{4.7}$$

[2] If more than one exists.
[3] The purpose of the pre-factor $1/\epsilon$ is so that the integral of U does not vanish as $\epsilon \to 0$.

where $\Omega = c_0\,\kappa + \epsilon^2\,\frac{1}{6}\,\mu_0\,\kappa^3 + ...$, and a is defined by the equation. Clearly: a is the solution by Fourier Transform of (4.5). Finally, in terms of χ and τ the amplitude satisfies the linear KdV equation

$$a_\tau - \frac{1}{6}\,\mu_0\,a_{\chi\chi\chi} = 0. \tag{4.8}$$

5 Radiation damping #02

5.1 Statement: Radiation damping #02

Consider a semi-infinite shallow water channel, with a rectangular cross-section of width $\boldsymbol{w}$, and a paddle at the end. The paddle has mass $\boldsymbol{m}$, and it is kept in place by a spring — with spring constant $\boldsymbol{\kappa}$. In addition, a force $\boldsymbol{f = f(t)}$ is applied by the paddle. We model the system using the shallow water wave equations in the channel

$$h_t + (h\,u)_x = 0 \quad \text{for} \quad x > \sigma(t), \tag{5.1}$$

$$u_t + u\,u_x + g\,h_x = 0 \quad \text{for} \quad x > \sigma(t), \tag{5.2}$$

where $h = h(x,\,t)$ is the water depth, $u = u(x,\,t)$ is the flow velocity, $\boldsymbol{g}$ is the acceleration of gravity, and $x = \sigma(t)$ is the position of the paddle. At the paddle position, $x = \sigma$, we have

$$\dot{\sigma} \;=\; u, \tag{5.3}$$

$$m\,\ddot{\sigma} \;=\; -\kappa\,(\sigma - x_0) - \underbrace{\frac{1}{2}\,g\,\rho\,w\,h^2}_{p_w} + f, \tag{5.4}$$

where $\boldsymbol{\rho}$ is the density of water, p_w is the pressure[4] force by the water on the paddle, and x_0 corresponds to the equilibrium position for the spring — see (5.5). Note that a forcing done by moving the spring attachment point, so that the spring force has the form $-\kappa\,(\sigma - x_0 - \chi(t))$, is the particular case of (5.4) where $f = \kappa\,\chi$ — in fact, we can always write f in this form.

At equilibrium

$$\sigma = 0, \quad u = 0, \quad h = H = \text{constant}, \quad f = 0, \quad \text{and} \quad \kappa\,x_0 = \frac{1}{2}\,g\,\rho\,w\,H^2. \tag{5.5}$$

Assume now an "infinitesimal" force and "infinitesimal" perturbations from equilibrium,[5] with $h = H + \eta$, and **write linearized equations of motion for $\boldsymbol{\sigma}$, $\boldsymbol{\eta}$, and $\boldsymbol{u}$.** The equations for η and u will apply for $x > 0$, with boundary conditions at $x = 0$ involving both f and σ. Introduce the velocity potential ϕ, with $u = \phi_x$ and $\eta = -\frac{1}{g}\,\phi_t$, and **write the equations in terms of $\boldsymbol{\phi}$ and $\boldsymbol{\sigma}$.**

For the situation where all the transient waves in the channel are gone, and the waves there are solely the product of the forcing f, **derive an ode for $\boldsymbol{\sigma}$.** This ode will have a damping coefficient, due to radiated energy carried away by the waves. Finally, **write the solution for the case when $\boldsymbol{f = a\,e^{i\,\omega\,t}}$ — where $\boldsymbol{a}$ and $\boldsymbol{\omega}$ are constants.**

5.2 Answer: Radiation damping #02

The linearized equations are

$$\eta_t + H\,u_x = 0, \quad \text{and} \quad u_t + g\,\eta_x = 0, \tag{5.6}$$

valid for $x > 0$. The boundary conditions at $x = 0$ follow from (5.3 – 5.4)

$$\dot{\sigma} = u, \quad \text{and} \quad m\,\ddot{\sigma} + \kappa\,\sigma = -g\,\rho\,w\,H\,\eta + f. \tag{5.7}$$

[4] Hydrostatic equilibrium, with pressure variations in the air above the water neglected.
[5] That is: u, σ, and η are "infinitesimal."

In terms of the velocity potential ϕ, such that $u = \phi_x$ and $\eta = -\dfrac{1}{g}\,\phi_t$, the second equation in (5.6) is automatically satisfied, and the system reduces to

$$\begin{aligned} \phi_{tt} - c^2\,\phi_{xx} &= 0 && \text{for} \quad x > 0, && (5.8)\\ \dot{\sigma} &= \phi_x && \text{at} \quad x = 0, && (5.9)\\ m\,\ddot{\sigma} + \kappa\,\sigma &= \rho\,w\,H\,\phi_t + f && \text{at} \quad x = 0, && (5.10) \end{aligned}$$

where $c = \sqrt{g\,H}$.

After the transient waves are gone, the solution must have the form

$$\phi = y\left(t - \frac{x}{c}\right), \tag{5.11}$$

for some function of a single variable $y = y(t)$. Substituting this into (5.9 – 5.10) yields the ode

$$m\,\ddot{\sigma} - \rho\,w\,H\,c\,\dot{\sigma} + \kappa\,\sigma = f, \quad \text{where} \quad y = -c\,\sigma + \text{constant} \tag{5.12}$$

and $\boldsymbol{\nu_R = \rho\,w\,H\,c}$ is the **damping coefficient.** The constant in the expression for y has no physical relevance, since it has no effect on η or u.

If $f = a\,e^{i\,\omega\,t}$, then

$$\sigma = \frac{a}{\kappa - m\,\omega^2 - i\,\omega\,\nu_R}\,e^{i\,\omega\,t}, \quad y = -c\,\sigma, \quad \text{and } \phi \text{ follows from (5.11).} \tag{5.13}$$

Important point. *The homogeneous solutions to (5.12) are exponentially damped. If these solutions are plugged into (5.11), they yield solutions that blow up exponentially as $x \to \infty$. This is absurd, how can it be? The problem arises because the homogeneous solutions to (5.12) are transients. We cannot assume that they have existed for all times, as this requires infinite energy as $t \to -\infty$. On the other hand, if such a solution is initiated at a finite time t_0, and vanishes before the starting time t_0, then (5.11) yields a reasonable answer, with no waves beyond $x = c\,(t - t_0)$.*

6 Reflected wave from an active boundary #01

6.1 Statement: Reflected wave from an active boundary #01

Calculate the reflected wave for the linear wave equation problem in figure 6.1. Note that:

1. *There are no waves on $x < 0$. The equation applies for $x > 0$ only.*
2. *There is a critical angle θ_c at which something very special happens.*
  Find θ_c, and explain the physical meaning of what you found.
3. Show also that the model is stable: *none of the normal modes grows* — see remark **6.1**.

The equation to be solved is

$$\Phi_{tt} - c^2\,\Delta\,\Phi = 0 \;\text{ for }\; x > 0, \quad \text{where } \; c > 0 \; \text{ is a constant,} \tag{6.1}$$

with the boundary condition

$$\Phi_{xt} + \gamma^2\,\Phi_{yy} = 0 \;\text{ at }\; x = 0, \quad \text{where } \; \gamma > 0 \; \text{ is a constant.} \tag{6.2}$$

This is a made-up *mathematical model for something that happens when there is an "active" boundary* — i.e.: extra energy is available there. The motivating example here is combustion, where the boundary is a plane detonation wave (where a chemical reaction occurs) and we look at the linearized problem near this solution. The "real" problem is far more complicated than the model here, with more waves (here only the acoustic waves have been allowed to survive), and more variables. But the basic wave phenomena that occurs is the same that you will find here.

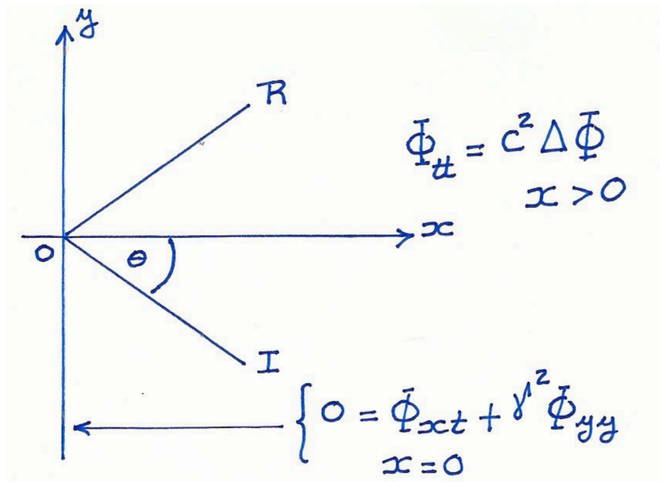


Figure 6.1: Reflected wave from an active boundary. Here $c,\,\gamma > 0$ are constants. The incident wave has the form $\boldsymbol{I = \exp\left(i\,k\,(-x\,\cos\theta + y\,\sin\theta - c\,t)\right)}$, with $0 < \theta < \pi/2$ and $k \neq 0$ constants.

Remark 6.1 *Because the equation and boundary condition are translational invariant in the y-direction, the problem can be Fourier Transformed in this variable, and we can write*

$$\Phi = \int_{-\infty}^{\infty} \phi(\ell,\,x,\,t)\,e^{i\,\ell\,y}\,d\ell, \quad \textit{where} \quad \phi = \frac{1}{2\,\pi}\int_{-\infty}^{\infty} \Phi\,e^{-i\,\ell\,y}\,dy \tag{6.3}$$

satisfies $$\phi_{tt} - c^2\,\phi_{xx} + \ell^2\,c^2\,\phi = 0 \quad \textit{for} \quad x > 0, \tag{6.4}$$
and $$\phi_{xt} - \gamma^2\,\ell^2\,\phi = 0 \quad \textit{at} \quad x = 0, \tag{6.5}$$
where $-\infty < \ell < \infty$.
The normal modes are solutions of this problem of the form $\phi = \varphi(\ell,\,x)\,e^{\lambda\,t}$, *decaying as* $x \to \infty$, *where* λ *is a constant. Since the equation for* φ *is a constant coefficients ode, it must be* $\varphi \propto e^{-\alpha\,x}$, *where* α *is a constant such that* $Re(\alpha) > 0$. ♣

6.2 Answer: Reflected wave from an active boundary #01

The solution that we need to compute has the form

$$\Phi = \exp\left(i\,k\,(-x\,\cos\theta + y\,\sin\theta - c\,t)\right) + R\,\exp\left(i\,k\,(x\,\cos\theta + y\,\sin\theta - c\,t)\right), \tag{6.6}$$

where R is the reflection coefficient. By construction this is a solution to the wave equation (6.1). Substitution into the boundary condition (6.2) yields

$$(c\,\cos\theta + \gamma^2\,\sin^2\theta) - (c\,\cos\theta - \gamma^2\,\sin^2\theta)\,R = 0 \tag{6.7}$$

(note that this equation is independent of k*).*[6] Hence $$R = \frac{c\,\cos\theta + \gamma^2\,\sin^2\theta}{c\,\cos\theta - \gamma^2\,\sin^2\theta}. \tag{6.8}$$

Notice now that **there is a critical angle $\boldsymbol{\theta_c}$**, defined by[7] $$c\,\cos\theta_c = \gamma^2\,\sin^2\theta_c. \tag{6.9}$$
At the critical angle $$R(\cos\theta_c) = \infty. \tag{6.10}$$

The *infinity in the reflection coefficient corresponds* to the fact that **spontaneous emission of radiation** by the active boundary, at the critical angle, is possible. That is

$$\Phi = \exp\left(i\,k\,(x\,\cos\theta_c + y\,\sin\theta_c - c\,t)\right), \tag{6.11}$$

[6] Not surprising, since the wave equation has no dispersion.
[7] See figure 6.2.

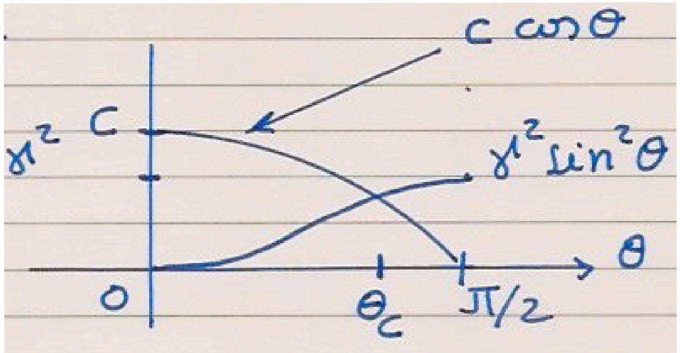


Figure 6.2: Critical angle, $0 < \theta_c < \pi/2$, defined by the intersection of the curves $c\,\cos\theta$ and $\gamma^2\,\sin^2\theta$.

is a solution to (6.1–6.2), for any k, because of (6.9). In other words, *a "reflected wave" can occur without the need of an incident wave!*

Does this mean that this problem is un-physical? Does the system have solutions that extract energy out of the boundary, and grow without bounds? To ascertain this issue, we investigate the "normal modes" for the equation. Substituting $\boldsymbol{\phi = e^{-\alpha\,x+\lambda\,t}}$ into (6.4–6.5), with α and λ constants to be found ($\mathrm{Re}(\alpha) > 0$), leads to

$$\lambda^2 - c^2\,\alpha^2 + c^2\,\ell^2 = 0 \quad \text{and} \quad \alpha\,\lambda + \gamma^2\,\ell^2 = 0. \tag{6.12}$$

Note that, for $\ell = 0$, the only solution of these equations is $\alpha = \lambda = 0$, which does not satisfy $\mathrm{Re}(\alpha) > 0$. Thus: *without loss of generality, we assume* $\ell \neq 0$. (6.13)

Next we show stability: $\mathrm{Re}(\lambda) < 0$, *all the normal modes decay.* (6.14)

Proof: we can write $\alpha = \rho\,e^{i\,\psi}$, where $\rho > 0$ and $-\pi/2 < \psi < \pi/2$. Hence, from the second equation in (6.12), $\lambda = -(\gamma^2\,\ell^2/\rho)\,e^{-i\,\psi}$, from which it follows that $\mathrm{Re}(\lambda) < 0$. ♣

The equations in (6.12) can be solved explicitly, as follows: Multiply the first equation by α^2, and then use the second to eliminate $\lambda^2\,\alpha^2$. This gives

$$c^2\,\alpha^4 - c^2\,\ell^2\,\alpha^2 - \gamma^4\,\ell^4 = 0. \tag{6.15}$$

Thus

$$\alpha = |\ell|\,\sqrt{\frac{c + \sqrt{c^2 + 4\,\gamma^4}}{2\,c}} > 0 \quad \text{and} \quad \lambda = -2\,|\ell|\,\sqrt{\frac{-c + \sqrt{c^2 + 4\,\gamma^4}}{2\,c}} < 0, \tag{6.16}$$

where the other three roots are excluded because they give α either pure imaginary,[8] or negative. Hence, in fact, **both α and λ are real.**

Remark 6.2 *It is easy to see that the purely imaginary roots of (6.15) correspond to the solution in (6.11), and the one obtained by the symmetry $y \to -y$ — equivalently: $\theta_c \to -\theta_c$.*

To see this, take one of the two solutions with α pure imaginary. For example, assume that $\ell > 0$, and take $\alpha = -i\,\ell\,\sqrt{\left(\sqrt{c^2 + 4\,\gamma^4} - c\right)/(2\,c)}$.
Write $\ell = k\,\sin\theta$ and $\alpha = -i\,k\,\cos\theta$, where $0 < \theta < \pi/2$ and $0 < k = \sqrt{\ell^2 - \alpha^2}$. Then (6.15) yields

$$0 = c^2\,\cos^2\theta - \gamma^4\,\sin^4\theta = (c\,\cos\theta - \gamma^2\,\sin^2\theta)\,(c\,\cos\theta + \gamma^2\,\sin^2\theta).$$

Since $\cos\theta > 0$, this is (6.9). The other cases can be done in the same fashion. ♣

Remark 6.3 *To have a complete argument that shows that there is nothing wrong with the model in (6.1–6.2), we would need to show that the initial value problem for the system behaves properly. This can be done by, for example, using the Laplace transform to produce the solution to (6.4–6.5) — hence, via (6.3) the solution to (6.1–6.2). This is left as a challenge to the reader.* ♣

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[8] See remark **6.2**.

7 Shallow water waves at the beach

7.1 Statement: Shallow water waves at the beach

Modulation theory for a slowly varying linear wave in a dispersive media states that the wave-number $\vec{k} = \nabla\,\theta$, and the wave-frequency $\omega = -\theta_t$, must satisfy the dispersion relation

$$G(\omega,\, \vec{k}) = 0, \tag{7.1}$$

where θ is the wave-phase. The wave-amplitude, in turn, satisfies the energy equation [9]

$$(a^2)_t + \mathrm{div}(\vec{c}_g\, a^2) = 0, \tag{7.2}$$

where $\vec{c}_g = \nabla_{\vec{k}}\, \omega$ is the group speed.

Example 1: For the Klein-Gordon equation $u_{tt} - c^2\,\Delta\, u + m^2\, u = 0$, where $c > 0$ and $m > 0$ are constants, $G = \omega^2 - c^2\, k^2 - m^2$ and $\vec{c}_g = (c^2/\omega)\,\vec{k}$. The theory works even if the waves are not dispersive, as in the case of linear shallow water

$$G = \omega^2 - g\,h\,k^2 \quad \text{and} \quad \vec{c}_g = (g\,h/\omega)\,\vec{k}, \tag{7.3}$$

where $h > 0$ is the water depth. ♣

If the media is slowly varying in space (changes happen on scales much larger than the wave-length) the theory still applies, with (7.1) replaced by

$$G(\omega,\, \vec{k},\, \vec{x}) = 0, \tag{7.4}$$

and $\vec{c}_g = \vec{c}_g(\vec{k},\, \vec{x})$ in (7.2).

Example 2: Take $h = h(\vec{x})$ in (7.3). ♣

In particular, for single frequency waves, we can take $\theta = \phi(\vec{x}) - \omega\, t$, and $a = a(\vec{x})$, where ω is a constant. Then

$$G(\omega,\, \vec{k},\, \vec{x}) = 0 \quad \text{and} \quad \mathrm{div}(\vec{c}_g\, a^2) = 0, \tag{7.5}$$

where $\vec{k} = \nabla\,\phi$.

Problem tasks/questions: Consider the case of shallow water, in 1-D, for the situation where single frequency wave is approaching the shore at a gently sloped beach (thus $h = h(x)$ approaches zero slowly). Answer the following questions:

A. How do the wave-number and wave-length behave as h vanishes?
B. How does the wave-amplitude behave as h vanishes?
C. How does the maximum wave slope behave as h vanishes?
D. Does the linear approximation remain valid all the way to the shore?

7.2 Answer: Shallow water waves at the beach.

The equations are

$$\omega^2 = g\,h\,k^2 \quad \text{and} \quad (\sqrt{g\,h}\,a^2)_x = 0, \quad \text{where } k = \phi_x. \tag{7.6}$$

Hence

$$k = O(h^{-1/2}), \quad \lambda = O(h^{1/2}), \quad a = O(h^{-1/4}), \quad \text{and} \quad s = a\,k = O(h^{-3/4}), \tag{7.7}$$

where s is the wave slope. [10] It follows that the wave-length shrinks, and the amplitude grows, as the wave approaches the shore. Eventually the linear approximation must break down — what happens then is that the nonlinearity makes the waves break.

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[9] The wave-energy flows at the group speed. Note that here we assume $a > 0$.
[10] Let $\eta = a\,\sin(\theta)$ be the wave height. Then the slope is $\eta = a\,k\,\cos(\theta)$.

8 Slowly varying harmonic signaling for a string on a bed

8.1 Statement: Slowly varying harmonic signaling for a string on a bed

The equation for the (linear) vibrations of an homogeneous string under tension, over an homogeneous elastic bed, has the form

$$\rho\, u_{tt} - T\, u_{xx} + \kappa\, u = 0, \tag{8.1}$$

where $u = u(x,\, t)$ is the deviation from equilibrium of the string, and the (positive) constants ρ, T, and κ are the string density, the string tension, and the bed elastic constant, respectively.

In the lectures we analyzed the signaling problem, for a semi-infinite string $0 < x < \infty$, characterized by the boundary condition

$$u(0,\, t) = \mathrm{Re}\left(a\, e^{-i\,\Omega\, t}\right), \tag{8.2}$$

where a is a complex constant, and $\Omega > 0$ is a real constant. If $\boldsymbol{\Omega > \omega_c = \sqrt{\kappa/\rho}}$, this signaling problem has the general "steady state" solution

$$u(x,\, t) = \mathrm{Re}\left((a-b)\, e^{i\,(K\,x - \Omega\, t)} + b\, e^{i\,(-K\,x-\Omega\, t)}\right), \quad \text{with } \; K = \sqrt{(\rho/T)(\Omega^2 - \omega_c^2)} > 0, \tag{8.3}$$

where b is a constant. By calculating the average (over one time period $2\,\pi/\Omega$) energy flux produced by a plane harmonic wave — namely: $< -T\, u_t\, u_x >$ — we showed that **it must be $\boldsymbol{b = 0}$,** since the component $\boldsymbol{b\, e^{i\,(-K\,x - \Omega\, t)}}$ in (8.3) corresponds to an energy flux from infinity towards $x = 0$.

This problem aims at arriving to the same result, but using a different approach. (8.4)

We begin by considering the a-dimensional version of the problem above, namely:

$$u_{tt} - u_{xx} + u = 0, \quad \text{for } \; x > 0, \tag{8.5}$$

with boundary condition

$$u(0,\, t) = \mathrm{Re}\left(a\, e^{-i\,\Omega\, t}\right), \quad \text{where } \; \Omega > 1 = \omega_c. \tag{8.6}$$

The corresponding wave number is then $\boldsymbol{K = \sqrt{\Omega^2 - 1}}$.

Next we generalize the problem to a situation where the forcing at the starting end of the string is **turned on slowly.** Namely, instead of taking a constant in (8.6), we assume that a is a slow[11] function of time

$$a = a(\tau), \quad \text{where } \; \tau = \epsilon\, t \quad \text{and } \; 0 < \epsilon \ll 1. \tag{8.7}$$

Because the amplitude $a(\tau)$ varies slowly, we expect that the solution to (8.5 – 8.7) will be, at least at leading order, close to the solution to the steady state that occurs when a is a constant — hence, it will have the same form. On the other hand, because the forcing amplitude is not constant, we expect that the amplitude of the solution to (8.5 – 8.7) will change slowly. Furthermore, the speed of propagation of changes in the forcing is bounded,[12] hence the amplitude of the solution to (8.5 – 8.7) cannot be independent of x, as this would require infinite speed of propagation. But, because the forcing varies slowly, a finite speed of propagation suggests that the space dependence should also be slow, as the amplitude has time to adjust (anywhere) to some sort of local steady state amplitude.

The arguments in the prior paragraph are somewhat vague, but they suggest that we should look for solutions of (8.5 – 8.7) of the form

$$\boldsymbol{u \approx \mathrm{Re}\left(A(\chi,\, \tau)\, e^{i\,(K\,x - \Omega\, t)}\right)}, \quad \text{where } \; \boldsymbol{\chi = \epsilon\, x.} \tag{8.8}$$

TASK #1: Seek such solutions, and **find the equation that $\boldsymbol{A = A(\chi,\, \tau)}$ must satisfy.**

Hint. *The idea is that the right hand side in (8.8) should be a solution up to some small error. Hence write*

$$u = \mathrm{Re}\left(A(\chi,\, \tau)\, e^{i\,(K\,x - \Omega\, t)}\right) + \epsilon\, u_1(x,\, t;\, \chi,\, \tau) + \dots \tag{8.9}$$

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[11] The reason we **need to use a-dimensional variables** in this problem is that, otherwise, "small" has no meaning.

[12] (8.5) is hyperbolic, with characteristic speeds ± 1. It can be shown that no signal propagates faster than 1.

substitute into the equation, and find an equation that u_1 must satisfy as a function of x and t. This equation will be forced by terms produced by the χ and τ dependence of A. Now select this dependence in such a way that no growing component in u_1 is triggered by the forcing — if u_1 grows, then the error in the approximate solution in (8.8) will not be small, so that (8.8) will not really be an approximate solution.

TASK #2: Perform the same analysis, but now look for solutions of (8.5 – 8.7) of the form

$$u \approx \mathrm{Re}\left(B(\chi,\,\tau)\,e^{i\,(-K\,x-\Omega\,t)}\right), \tag{8.10}$$

and find the equation that B satisfies.

TASK #3: The equations that you find for A and B will be fairly simple, and you will be able to write their general solution explicitly. By looking at these solutions, *argue that (8.9) is an acceptable solution to the problem in (8.5 – 8.7), but (8.10) is not.* This is the result promised in (8.4).

TASK #4: What role does the group speed play in all of this? Where does it appear?

8.2 Answer: Slowly varying harmonic signaling for a string on a bed

Substitute

$$u = \mathcal{A}(\chi,\,\tau)\,e^{i\,(k\,x-\omega\,t)} + \underbrace{\epsilon\,u_1(x,\,t;\,\chi,\,\tau) + \ldots}_{\text{error } \mathcal{E}}, \tag{8.11}$$

where k and ω are (real) constants satisfying $\omega^2 = 1 + k^2$, into (8.5). This yields the following leading order equation for the error

$$(u_1)_{tt} - (u_1)_{xx} + u_1 = 2\,i\,\underbrace{(\omega\,\mathcal{A}_\tau + k\,\mathcal{A}_\chi)}_{C}\,e^{i\,(k\,x-\omega\,t)} + O(\epsilon), \tag{8.12}$$

where C is independent of x and t, hence a constant as far as the operator on the left for u_1. Thus [13]

$$u_1 = -\frac{t}{\omega}\,(\omega\,\mathcal{A}_\tau + k\,\mathcal{A}_\chi)\,e^{i\,(k\,x-\omega\,t)} + U_1, \tag{8.13}$$

where U_1 is a solution to the homogeneous equation. However, in order for the first term in (8.11) to be a good approximate solution to the equation, the error should remain small as t grows. Hence we argue that $\mathcal{A}$ should satisfy the equation

$$\omega\,\mathcal{A}_\tau + k\,\mathcal{A}_\chi = 0 \quad\Longleftrightarrow\quad \mathcal{A}_\tau + c_g\,\mathcal{A}_\chi = 0, \tag{8.14}$$

where $c_g = \dfrac{k}{\omega} = \dfrac{d\omega}{dk}$ is the group speed.

TASKS #1 and #2. From (8.14), the equations for A in (8.9) and for B in (8.10) are

$$A_\tau + c\,A_\chi = 0 \quad\text{and}\quad B_\tau - c\,B_\chi = 0, \quad\text{where}\quad c = \frac{K}{\Omega} > 0. \tag{8.15}$$

TASK #3. The solutions to (8.15) that correspond to the boundary condition in (8.7) are

$$A = a\left(\tau - \frac{\chi}{c}\right) \quad\text{and}\quad B = a\left(\tau + \frac{\chi}{c}\right). \tag{8.16}$$

The solution for A shows the wave amplitude changes propagating from the origin to the right, while the solution for B corresponds to information moving from infinity towards the origin. Hence (8.9) is an acceptable solution, while (8.10) is not.

TASK #4. From equation (8.14), it should be obvious that the information about changes in the wave amplitude (hence energy) propagates at the group speed.

[13] In calculating u_1, we neglect the $O(\epsilon)$ in (8.12), with the argument that it will be absorbed into the calculation of the next term in (8.11); i.e.: $\epsilon^2\,u_2$.

9 String on an elastic bed: harmonic forcing at the critical frequency

Before doing this problem, check the *Lecture topics for 18376* notes: Section: Radiation damping. Subsection: Semi-infinite string over elastic bed with mass-spring at end. Subsubsection: Harmonic forcing.

9.1 Statement: String on an elastic bed: harmonic forcing at the critical frequency

Consider a semi-infinite string over an elastic bed, under tension, with a forced mass-spring system attached at its end (assume also small, in-plane, motion). With properly selected a-dimensional variables, the equations are

$$u_{tt} - u_{xx} + u = 0 \qquad \text{for } x > 0, \tag{9.1}$$
$$u_{tt} + \Omega^2 u = 2\nu u_x + G \qquad \text{at } x = 0, \tag{9.2}$$

where Ω and ν are positive constants, $G = G(t)$ is the force applied to the mass attached to the string, and $2\nu u_x(0,\, t)$ is the force by the string (due to its tension) on the mass. We will make the following assumptions:

a1. The forcing is harmonic, specifically: $\boldsymbol{G = e^{i\,\omega\,t}}$, with $\boldsymbol{0 < \omega < 1}$. (9.3)

a2. The following applies: $\boldsymbol{\Omega^2 < 1}$. (9.4)

A particular solution to (9.1–9.3) is given by $\boldsymbol{u_p = a\, e^{i\,\omega\,t - \ell\,x}}$, with $\boldsymbol{a = 1/(\Omega^2 + 2\nu\ell - \omega^2)}$, (9.5)
where $\boldsymbol{\ell = \sqrt{1-\omega^2}}$.
Below we show how to use this solution to generate the solution to the initial value problem for (9.1–9.3). However, note: **there is a critical value of ω, ω_c, at which (9.5) fails** — the value such that $\Omega^2 + 2\nu\ell - \omega^2 = 0$.

Remark 9.1 $\Omega^2 + 2\nu\ell - \omega^2 = 0$ has exactly one solution for $0 < \omega < 1$, ω_c.
Proof. Let $f(\omega) = \omega^2 - 2\nu\ell$. This function is increasing, and satisfies $f(0) = -2\nu < \Omega^2$ and $f(1) = 1 > \Omega^2$. ♣

Problem task: Find a particular solution for the case $\omega = \omega_c$.

Hint. *Use the technique illustrated by the following example:* Consider the ode: $\boldsymbol{\ddot{y} + y = e^{i\,\omega\,t}}$ [A]. This has the solution $y_p = (1-\omega^2)^{-1} e^{i\,\omega\,t}$, valid as long as $\omega^2 \neq 1$. To find a solution for $\omega = 1$, notice that $z = e^{i\,\omega\,t}$ satisfies, for <u>any</u> ω, the ode: $\ddot{z} + z = (1-\omega^2)\, e^{i\,\omega\,t}$ [B]. Now let $\xi = \frac{\partial z}{\partial \omega}$, and take the derivative of [B] with respect to ω. This yields the equation: $\ddot{\xi} + \xi = -2\,\omega\, e^{i\,\omega\,t} + i\,t\,(1-\omega^2)\, e^{i\,\omega\,t}$ [C]. Evaluating now [C] at $\omega = 1$ leads to the desired particular solution, specifically: $\boldsymbol{y_p = -\frac{1}{2}\,\xi(\omega = 1) = -\frac{1}{2}\, i\, t\, e^{it}}$.
Note also that the answer to this problem is just slightly longer than this hint. ♣

From particular to the general solution.

Here we show how to use a particular solution of (9.1–9.3), to reduce the initial value problem to one that can be solved by ode techniques and Fourier Transforms. Note: you do not need to read this to do the problem!
Disclaimer: the approach presented below is probably not *"the best"*. Think of it as a *proof of concept* only.

We begin by writing $\boldsymbol{u = u_p + w}$, where **w solves (9.1–9.2) with $G = 0$ and initial data:**

$$\boldsymbol{w(x,\,0) = w_0(x) = u(x,\,0) - u_p(x,\,0)} \text{ and } \boldsymbol{w_t(x,\,0) = w_1(x) = u_t(x,\,0) - (u_p)_t(x,\,0)}. \tag{9.6}$$

Introduce now $v = v(x,\,t)$ by where the operator $\mathcal{L}$ is defined by the equation. Then

$$\boldsymbol{v = \mathcal{L}\, w = w_{xx} - (1-\Omega^2)\, w - 2\,\nu\, w_x}, \tag{9.7}$$

$$\boldsymbol{v_{tt} - v_{xx} + v = 0} \quad \text{for } x > 0, \text{ with } \boldsymbol{v(0,\,t) = 0}. \tag{9.8}$$

The initial conditions for this equation $\Big(\boldsymbol{v(x,\,0) = v_0(x)}$ and $\boldsymbol{v_t(x,\,0) = v_1(x)}\Big)$ follow from (9.6–9.7).
Why is it $v(0,\,t) = 0$? This is because w satisfies (9.1), so that $\boldsymbol{v = w_{tt} + \Omega^2\, w - 2\,\nu\, w_x}$ as well.
Then

$$v = \int_0^\infty \left(\hat{v}_0(k)\, \sin(k\,x)\, \cos\left(\sqrt{1+k^2}\, t\right) + \hat{v}_1(k)\, \sin(k\,x)\, \frac{\sin\left(\sqrt{1+k^2}\, t\right)}{\sqrt{1+k^2}} \right) \mathrm{d}k, \tag{9.9}$$

where $\hat{v}_0$ and $\hat{v}_1$ are the sine-Fourier Transforms of v_0 and v_1.

The issue is now: **Given v, how do we recover w?** To do this we observe that, from the definition of v, we have $\boldsymbol{\mathcal{L}\, w = v}$. (9.10)
Thus

$$w(x,\,t) = w_1(x,\,t) + \alpha(t)\, e^{\lambda_1\, x}, \quad \text{where} \quad w_1(x,\,t) = \int_0^\infty G(x,\,y)\, v(y,\,t)\, \mathrm{d}y, \tag{9.11}$$

α is a function to be determined, λ_1 is defined below, and G is the Green's function for $\mathcal{L}^{-1}$ with zero boundary condition at $x = 0$. That is:

$$G = \frac{1}{\lambda_1 - \lambda_2}\left(e^{h\,(x-y)} - e^{\lambda_1 x - \lambda_2\, y}\right) \tag{9.12}$$

where $h = \lambda_2$ if $x < y$, $h = \lambda_1$ if $x > y$, and

$$\lambda_1 = -\nu - \sqrt{\nu^2 + (1 - \Omega^2)} < 0,$$
$$\lambda_2 = -\nu + \sqrt{\nu^2 + (1 - \Omega^2)} > 0.$$

The λ_j are the two roots of $\lambda^2 - 2\,\nu\,\lambda = 1 - \Omega^2$, the characteristic equation for $\mathcal{L}$.

Why (9.11)? Because $\mathcal{L}\,w = v$ determines w up to an homogeneous solution, but $e^{\lambda_2\,x}$ is not allowed because $\lambda_2 > 0$.

Now, because v satisfies (9.1), and $\mathcal{L}\,w_1 = v$, we have $\mathcal{L}((w_1)_{tt} - (w_1)_{xx} + w_1) = 0$, hence

$$(w_1)_{tt} - (w_1)_{xx} + w_1 = \beta(t)\,e^{\lambda_1\,x}. \tag{9.13}$$

But both v and w_1 vanish at $x = 0$. Hence evaluating (9.10) and (9.13) at $x = 0$ we obtain: $(w_1)_{xx} = 2\,\nu\,(w_1)_x$ and $(w_1)_{xx} = -\beta$. Thus

$$\boldsymbol{\beta(t) = -2\,\nu\,(w_1)_x(0,\,t)}. \tag{9.14}$$

Finally, substituting $w = w_1 + \alpha(t)\,e^{\lambda_1\,x}$ into $w_{tt} - w_{xx} + w = 0$, and using (9.13), yields an **equation that determines α.** That is

$$\boldsymbol{\ddot{\alpha} + (1 - \lambda_1^2)\,\alpha + \beta = 0}. \tag{9.15}$$

The task of finding how to get initial conditions for this equation is left to the reader. Note that w, as defined by all these steps, satisfies the boundary condition at $x = 0$. Why? Because using $w_{tt} - w_{xx} + w = 0$ in $\mathcal{L}\,w = v$ yields $v = w_{tt} + \Omega^2\,w - 2\,\nu\,w_x$, and v vanishes at $x = 0$.

Remark 9.2 Provided that the initial data are reasonably smooth: as $t \to \infty$, v vanishes; consequently, w as well. It follows that: **As $t \to \infty$, the solution to (9.1–9.3) is dominated by the particular solution, $u \sim u_p$.** ♣

Here you may wonder: *wait a second, the particular solution is not unique; what if I use a different one from the one above?* The answer is that it does not matter: the difference between any two particular solutions vanishes.

9.2 Answer: String on an elastic bed: harmonic forcing at the critical frequency

Let $\boldsymbol{z = e^{i\,\omega\,t - \ell\,x}}$, with $\boldsymbol{\ell}$ as in (9.5). Then $\boldsymbol{z}$ satisfies (9.1–9.2) with $\boldsymbol{G = (\Omega^2 + 2\,\nu\,\ell - \omega^2)\,e^{i\,\omega\,t}}$. Now take the derivative of these equations with respect to ω. This gives

$$\xi_{tt} - \xi_{xx} + \xi \;=\; 0 \qquad \text{for } x > 0, \tag{9.16}$$

$$\xi_{tt} + \Omega^2\,\xi \;=\; 2\,\nu\,\xi_x - \frac{2\,\omega}{\ell}\,(\nu + \ell)\,e^{i\,\omega\,t} + i\,t\,(\Omega^2 + 2\,\nu\,\ell - \omega^2)\,e^{i\,\omega\,t} \qquad \text{at } x = 0, \tag{9.17}$$

where $\boldsymbol{\xi = \partial\,z/\partial\,\omega = \left(i\,t + \frac{\omega}{\ell}\,x\right)\,z}$. Evaluating this at $\boldsymbol{\omega_c}$ shows that we can take

$$\boldsymbol{u_p = -\frac{\ell_c}{2\,\omega_c\,(\nu + \ell_c)}\left(i\,t + \frac{\omega_c}{\ell_c}\,x\right)\,e^{i\,\omega_c\,t - \ell_c\,x}}. \tag{9.18}$$

Note that, since $\omega^2 = 1 - \ell^2$, the critical equation is $0 = \Omega^2 + 2\,\nu\,\ell_c - \omega_c^2 = \Omega^2 + 2\,\nu\,\ell_c + \ell_c^2 - 1$, with $\ell_c > 0$. Thus $\boldsymbol{\ell_c = -\lambda_1}$, where λ_1 is defined in (9.12). **Hence, in (9.15), $\boldsymbol{1 - \lambda_1^2 = \omega_c^2}$.**

THE END.