Answers: 18.376 Problem Set #01, MIT Spring 2023

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1.1 Statement: The flux for a conserved quantity must be a vector

Note: Below (for simplicity) we present many arguments/questions in 2D. However they apply just as well in nD; $n = 3, 4, \ldots$

Consider some conserved quantity, with density $\rho = \rho(\vec{x}, t)$ and flux vector $\vec{q} = \vec{q}(\vec{x}, t)$ in 2D. Then, in the absence of sources or sinks, we made the argument that conservation leads to the (integral) equation

$$\frac{d}{dt}\int_{\Omega} \rho \, \mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\partial\Omega} \vec{q}\cdot\hat{n}\,\mathrm{d}s, \tag{1.1}$$

for any region Ω in the domain where the conserved "stuff" resides, where $\partial\Omega$ is the boundary of Ω, s is the arc-length along $\partial\Omega$, and $\hat{n}$ is the outside unit normal to $\partial\Omega$.

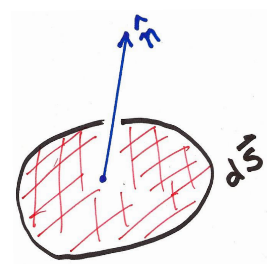
There are **two implicit assumptions** used above

1. The flux of conserved stuff is local: stuff does not vanish somewhere and re-appears elsewhere (this would not violate conservation). For most types of physical stuff this is reasonable. But one can think of situations where this is not true — e.g.: when you wire money, it disappears from your local bank account, and reappears elsewhere (with some loses due to fees, which go to other accounts).
2. The flux is given by a vector. But: $\Rightarrow$ *Why should this be so?* $\Leftarrow$ (1.2)
   The objective of this problem is to answer this question.

Given item **1**, **the flux can be characterized/defined as follows** †

> For any surface element $d\vec{S}$ (at a point $\vec{x}$, with unit normal $\hat{n}$) the flux indicates how much stuff, per unit time and unit area, crosses $d\vec{S}$ from one side to the other, in the direction of $\hat{n}$. (1.3)

† This is in 3D. For a 2D, change "surface element" to "line element".



It follows that the flux should be a scalar function of position, time, and direction. That is:

$$q = q(\vec{x},\, t,\, \hat{n}), \tag{1.4}$$

where q is the amount of stuff, per unit time and unit length, crossing a curve [1] with unit normal $\hat{n}$ from one side to the other (with direction [2] given by $\hat{n}$). Then (1.1) takes the form

$$\frac{d}{dt}\int_{\Omega} \rho \, \mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\partial\Omega} q(\vec{x},\, t,\, \hat{n})\,\mathrm{d}s. \tag{1.5}$$

However, in this form we cannot use Gauss' theorem to transform the integral on the right over $\partial\Omega$, into one over Ω. *This is a serious problem, for this is the crucial step in reducing* (1.1) *to a pde.*

Your task: *Show that,* provided that ρ and q are "nice enough" functions (e.g.: continuous partial derivatives), *equation* (1.5) *can be used to show that* q *has the form*

$$q = \hat{n}\cdot\vec{q}, \tag{1.6}$$

for some vector valued function $\vec{q}(\vec{x},\, t)$*.*

Hints.

[1] In 3D: "... and unit area, crossing a surface ..."

[2] A positive q means that the net flow is in the direction of $\hat{n}$.

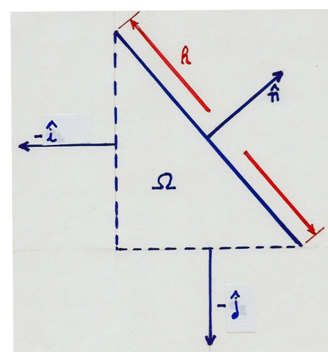
A. *It should be obvious that the flux going across any curve (surface in 3D) from one side to the other should be equal and of opposite sign to the flux in the opposite direction. That is, q in (1.4) satisfies*

$$q(\vec{x},\, t,\, -\hat{n}) = -q(\vec{x},\, t,\, \hat{n}). \tag{1.7}$$

Violation of this would result in the conserved "stuff" accumulating (or being depleted) at a finite rate from a region with zero area (zero volume in 3D), which is not compatible with the assumption that ρ_t is continuous and equation (1.5). **Note:** there are situations where it is reasonable to make models where conserved "stuff" can have a finite density on curves or surfaces (e.g.: surfactants at the interface between two liquids, surface electric charge, etc.). Dealing with situations like this requires a slightly generalized version of the ideas behind (1.7).

B. *Given an arbitrary small curve segment[3] of length $h > 0$ and unit normal $\hat{n}$, realize it as the hypotenuse of a right triangle where the other sides are parallel to the coordinate axes. Then write (1.5) for the triangle, divide the result by h, and take the limit $h \downarrow 0$. Note that, if the segment of length h is parallel to one of the coordinate axis, then one of the sides of the triangle has zero length, and the triangle has zero area — but the argument still works, albeit trivially (it reduces to the argument in* **A***).*

1.2 Answer: The flux for a conserved quantity must be a vector



Consider a small (straight) segment of length $0 < h \ll 1$, which is not parallel to the coordinate axes. Let $\hat{n}$ be the unit normal to the segment with a positive first component $n_1 > 0$. Assume that $n_2 > 0$. Construct a right triangle Ω, with two sides parallel to the coordinate axes and hypotenuse the given segment (see figure). Then $\partial\Omega$ has outside unit normals $\hat{n}$, $-\hat{\imath}$, and $-\hat{\jmath}$ (where $\hat{\imath}$ and $\hat{\jmath}$ are the coordinate axes unit vectors), and equation (1.5) yields

$$\begin{aligned} O(h^2) &= -q(\vec{x},\, t,\, \hat{n})\, h - q(\vec{x},\, t,\, -\hat{\imath})\, h\, n_1 - q(\vec{x},\, t,\, -\hat{\jmath})\, h\, n_2 + O(h^2), \\ &= -q(\vec{x},\, t,\, \hat{n})\, h + q(\vec{x},\, t,\, \hat{\imath})\, h\, n_1 + q(\vec{x},\, t,\, \hat{\jmath})\, h\, n_2 + O(h^2), \end{aligned} \tag{1.8}$$

where $\vec{x}$ is any point in Ω and we use (1.7) to obtain the second line from the first. Now divide (1.8) by h, and take the limit $h \downarrow 0$. This yields (1.7) with

$$\vec{q} = q(\vec{x},\, t,\, \hat{\imath})\, \hat{\imath} + q(\vec{x},\, t,\, \hat{\jmath})\, \hat{\jmath}. \tag{1.9}$$

To complete the proof we need to consider the cases:

— Case $n_1 > 0$ and $n_2 < 0$. The argument is exactly analogous to the one above.

— Case $n_1 < 0$ and $n_2 \neq 0$. Follows from the result in (1.7).

— Case $n_1 = 0$ or $n_2 = 0$. Trivial, given $\vec{q}$ as in (1.9), and (1.7).

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2 Small transversal vibrations of a beam

2.1 Statement: Small transversal vibrations of a beam

A beam is a structure where one dimension (the axial dimension) is much larger than the other two (the transversal dimensions), see item **c**. In this problem you are asked to derive an equation for the small transversal vibrations of an homogeneous elastic beam with (constant) rectangular cross section, which is not under tension or compression. Further simplifying assumptions are:

[3] You can assume it is straight, since a limit $h \downarrow 0$ will occur.

a. The wavelength of the vibrations is much bigger than the transversal dimensions of the beam.

b. The vibrations are in-plane. This means that the motion of the beam is restricted to the plane determined by its axis, and the direction of one of the sides of the rectangular cross section.
Think of a blade. When the two transversal dimensions are very different, it is much harder to excite vibrations along the larger direction.

Under the conditions stated above, the beam motion can be described in terms of the *position of the beam axis* $y = u(x, t)$. Let $\boldsymbol{\rho} =$ constant $> \mathbf{0}$ be the *mass per unit length of the beam.* Then:

Task #1 of 5. Use conservation of the transversal momentum to derive an equation for $\boldsymbol{u}$.

Task #2 of 5. Use conservation of energy to derive another equation for $\boldsymbol{u}$.

Task #3 of 5. Show that the solutions to the task #1 equation satisfy the task #2 equation.

To do the problem you will need a few things from § 2.1.1, as follows:

— To do task #1 you need (fo.8), where f_s is defined in item **5** — E and I are constants.

— In addition, to do task #2, you need (fo.6) and (fo.9), where τ_b is defined in item **6**.

— The summary of facts in (fo.10) may be useful.

You do not need anything else, but I strongly encourage you to read, and understand, § 2.1.1.

Hint for task #2. Do not forget that energy flow is not only produced by forces [force times velocity], but by torque as well [torque times angular velocity]

More tasks

Task #4 of 5. Show that the task #1 equation yields conservation of angular momentum.

Task #5 of 5. Take a thin steel blade with a rectangular cross section (e.g., the blade from a metal hand saw, see the picture). Clamp one end using a bench vise, and leave the other end free. For the equation in task #1, in this situation, what boundary conditions on u should be imposed at each end?
What if there is a frictionless-hinge at each end of the blade, that keeps the end fixed, but allows the blade to freely rotate there. What boundary conditions should be used in this case?

Hints for task #4. (1) Recall that the angular momentum of a mass point moving in a plane [4] is given by $A = \pm\, m\, v\, d$, where: m is the mass; v is the speed; d is the distance from the straight line through the point along the direction of motion, to some fixed point in space; and the sign is positive if the mass is moving counter-clockwise around the selected point in space. For example, if the point in space is the origin, and the point path is $x = a =$ constant and $y = v\, t$, then $A = m\, a\, v$. To do task #4, take the selected point to be the origin of coordinates. (2) Recall also that angular momentum is produced by torque, and that torque can manifest in two ways: "directly" (as in the torque applied through the axle to a wheel), or through forces (as when you rotate a wheel by pushing through the edge): a wheel-chair can either have a motor, or the user can move the wheels with his/her hands, or both (or someone else can push the chair).

Side remarks

c. In a string the transversal dimensions are neglected (thus a string has no bending resistance). On the other hand, for a beam they are assumed small, but their effect is not neglected (thus a beam can support a transversal load).

d. An elastic beam can support vibrations along each of the two transversal directions, as well as longitudinal vibrations. It can support torsional vibrations (twist along the axis) as well. In principle it can also support torsion along the transversal directions — but these are not consistent with the beam approximation (Euler-Bernoulli assumptions, see § 2.1.1).

[4] In general the angular momentum is a vector, but for in-plane motion it is a scalar.

Euler-Bernoulli beam theory states that the beam cross sections move as rigid planes when the beam vibrates, and remain normal to the beam axis (or, in 2-D, the beam center plane, in red here). Thus the beam motion can be described in terms of the axis behavior.

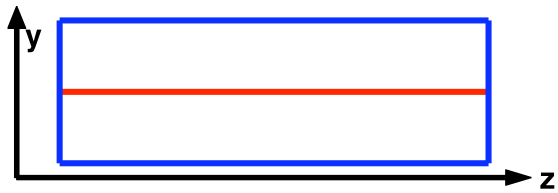

Figure 2.1: Cross section of a (rectangular) vibrating beam, of width w and height h.

The blue lines are the edge of the beam. The red line is the beam axis. The green lines are typical "fibers" — see paragraph above (fo.5). The magenta lines are typical beam cross sections, which move as rigid planes, and remain normal to the axis and fibers.

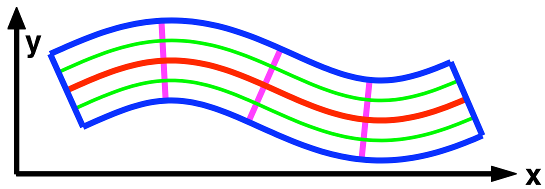

Figure 2.2: Side view of a beam undergoing in-plane motion, as per Euler-Bernoulli theory.

2.1.1 Beam elastic energy, shear force, and toque

This subsection's purpose is to provide contextual information needed to understand and do the problem. It does not include any further tasks to be done.

For beams under deformations that are not too large (basically, the situation in item **a** above), the Euler-Bernoulli assumptions apply [5] (see figures 2.1 and 2.2)

eb1. Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads, and can be assumed as rigid.

eb2. During deformation, the cross section of the beam remains planar, and normal to the deformed axis of the beam.

From this assumptions, and item **b**, it follows that we can describe the motion of the beam using just two 1D functions (see (fo.1) below, as follows:

1. Take a coordinate system such that the beam at equilibrium is $0 < x < L$, $|y| < \frac{1}{2}\,h$, and $|z| < \frac{1}{2}\,w$, where L is the beam length, h is its height, and w is its width. Here x is the axial coordinate, and the motion is in the x-y plane (no dependence on z).

2. We label each mass-element in the beam by its $(x,\, y,\, z)$ coordinates at equilibrium, and describe the beam at any time by giving the coordinates of each mass-element as a function of time and the equilibrium coordinates, that is: $X = X(x,\, y,\, t)$, $Y = Y(x,\, y,\, t)$, $Z = z$, where we have used item **b** to simplify the dependence on z.

3. Let $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x},\, \boldsymbol{t}) = \boldsymbol{X}(\boldsymbol{x},\, 0,\, \boldsymbol{t}) - \boldsymbol{x}$ and $\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{x},\, \boldsymbol{t}) = \boldsymbol{Y}(\boldsymbol{x},\, 0,\, \boldsymbol{t})$ be the two *functions describing the motion of the beam axis.* Then, from eb1 and eb2,

$$\left.\begin{array}{rcl} X & = & x + v - \frac{1}{d}\, y\, u_x, \\ Y & = & u + \frac{1}{d}\, y\,(1+v_x), \end{array}\right\} \text{ where } d = \sqrt{(1+v_x)^2 + u_x^2}. \tag{fo.1}$$

This follows because the unit normal vector to each planar cross section of the beam (which moves as a rigid body, as per eb1–eb2) is given by

$$\hat{n} = \frac{1}{d}\,(1+v_x,\, u_x,\, 0)^T \quad \Longrightarrow \quad \hat{t} = \frac{1}{d}\,(-u_x,\, 1+v_x,\, 0)^T. \tag{fo.2}$$

[5] These assumptions have been extensively confirmed for (solid cross section) slender beams made of isotropic materials.

Here $\hat{t}$ is the unit vector tangent to the planar cross section, in the $z = 0$ plane, pointing towards $y > 0$ (since $1 + v_x > 0$, because v_x is small, as explained below).

Detail: $\hat{n}$ is the unit tangent vector to the beam axis: $X = x + v$, $Y = v$, and $Z = 0$.

4. Finally, the condition of *small vibrations for a beam which is not under tension/compression* translates into

$$\text{Both } u_x \text{ and } v_x \text{ are small. Furthermore: } v_x = O(u_x^2). \tag{fo.3}$$

Thus *we neglect quadratic terms in u_x, and write*

$$X = x - y\, u_x \quad \text{and} \quad Y = y + u. \tag{fo.4}$$

The assumption here is that the deviations of the beam shape from horizontal and straight are small. Under this condition, in the absence of (significant) axial stretching or compression, the changes in horizontal dimensions of the beam cannot be larger than quadratic in u_x..

Because the beam cross sections do not deform, and behave as rigid surfaces,[6] all the deformation occurs along the curves $y =$ constant and $z =$ constant,[7] which are either stretched or compressed. We can thus obtain the *elastic energy* in the beam by computing the elastic energy in each fiber, and integrating over all of them. Along each fiber the change in arclength (relative to equilibrium) is

$$\Delta L\, ds = \left(\sqrt{X_x^2 + Y_x^2} - 1\right)\, dx = -y\, u_{xx}\, dx, \tag{fo.5}$$

upon use of (fo.3–fo.4). From remark **2.1**, $\frac{1}{2}\, E\, dz\, dy\, y^2\, \int u_{xx}^2\, dx$ is the elastic energy in each fiber. Thus

$$\mathcal{V} = \frac{1}{2}\, E\, I \int u_{xx}^2\, dx = \textbf{beam elastic energy}, \quad \text{where } I = \frac{1}{12}\, h^3\, w \tag{fo.6}$$

is the *second moment of the beam's cross section $I = \iint y^2\, dz\, dy$.*

Next we will use (fo.6) to compute both the **shear force, f_s,** and the **torque, τ_b,** along the beam:

5. At any point along the beam, $f_s = f_s(x,\, t)$ **is the force** in the y-direction (transversal) that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite force is applied by the right section on the left section). If the beam were to be cut, these are the forces that would be needed to keep in position the lips of the cut.

6. At any point along the beam, $\tau_b = \tau_b(x,\, t)$ **is the torque** that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite torque is applied by the right section on the left section). If the beam were to be cut, these are the torques that would be needed to keep the beam ends at the cut from rotating.

Let us now *investigate how $\mathcal{V}$ changes as we deform the beam, from some configuration to another.* To do this we need to apply forces to the beam, which do work against the opposite forces generated by the elastic deformation of the beam (this is how the energy changes). Thus, by looking at how the energy changes as the beam is deformed, we can ascertain the elastic forces in the beam for any given configuration.

Hence assume that $u = u(x,\, \tau)$, where τ is a parameter that we use to describe the successive configurations of the beam as its shape changes.[8] Then, for any arbitrary interval $a < x < b$, the energy within the interval varies as

[6] This is an approximation. There is some deformation and there are forces. But they are small and we neglect them.

[7] We will call these curves "fibers".

[8] The idea is that we deform the beam very, very, slowly. Thus, at every moment the applied forces, and the elastic forces generated by the deformation equilibrate each other exactly. There is no kinetic energy; all the work done by the applied forces go into the beam's elastic energy. Thus τ is a very "special" time.

follows

$$\begin{aligned}\frac{d\mathcal{V}}{d\tau} &= E\,I \int_a^b u_{xx}\,u_{xx\tau}\,dx \qquad \text{(integrate by parts twice)}\\ &= E\,I\,(u_{xx}\,u_{x\tau})\Big|_a^b - E\,I\,(u_{xxx}\,u_\tau)\Big|_a^b + E\,I\int_a^b u_{xxxx}\,u_\tau\,dx.\end{aligned} \qquad \text{(fo.7)}$$

This equation tells us that

7. The terms $E\,I\,u_{xxx}\,u_\tau$ at $x = a$, and $-E\,I\,u_{xxx}\,u_\tau$ at $x = b$, are the work (per unit τ-time) done by the applied forces to move the ends of the beam at the velocity u_τ of each end. This means that forces $E\,I\,u_{xxx}$ and $-E\,I\,u_{xxx}$ must be applied there. These must be the forces applied by the beam regions on each side of the interval. We conclude that $\boldsymbol{f_s = E\,I\,u_{xxx}}$. (fo.8)

8. Since $u_{x\tau}$ is the rate of change in beam angle, and work is also torque times angle, an argument entirely similar to the one in **7** shows that $\boldsymbol{\tau_b = -E\,I\,u_{xx}}$. (fo.9)

9. The last term in (fo.7) yields the force per unit length that must be applied along the beam to cause the deformation: $-E\,I\,u_{xxxx}$. The force per unit length needed to keep the beam with the given shape, and equilibrate the force $E\,I\,u_{xxxx}$ with which the beam pushes back.

Note that the forces producing beam deformation need not be "external forces" applied to the beam. When the beam is vibrating, the forces involved are those caused by the inertia of the beam.

Summary of useful facts. (fo.10)

$\boldsymbol{u}$ is the beam transversal deformation. The beam axis is $y = u(x, t)$ (small vibrations).
$\boldsymbol{u_t}$ is the beam transversal velocity.
$\boldsymbol{u_x}$ is the beam angle.
$\boldsymbol{u_{xt}}$ is the beam angular velocity.
$\boldsymbol{\tau_b}$ is the torque (see item **6**), given by $\tau_b = -E\,I\,u_{xx}$.
$\boldsymbol{f_s}$ is the shear force (see item **5**), given by $f_s = E\,I\,u_{xxx}$.
The elastic energy density is $\frac{1}{2}\,E\,I\,u_{xx}^2$.
I is the second moment of the beam's cross section. given by $I = \iint y^2\,dz\,dy$.

Remark 2.1 What is the energy stored in a slightly stretched/compressed elastic thin string?

The calculation below is valid as long as Hooke's law applies. In addition, we neglect any changes in the area of the string cross section when under tension or compression. In fact, the Euler-Bernoulli assumptions require that changes of this type be ignored.

First consider the energy stored in a short straight segment of length $L(t)$, as we stretch/compress it from length $L_0 = L(0)$ to $L_1 = L(T)$, where L_0 is the equilibrium length. From Hooke's law, the elastic force is

$$F = E\,\frac{L(t)-L_0}{L_0}\,A, \quad \text{where } A = \text{area of string cross section,} \qquad \text{(fo.11)}$$

and $\boldsymbol{E} =$ **Young's modulus** *(E has units of force over area). The energy at the end of the process is*

$$\mathcal{V}_{\text{seg}} = \int_0^T F(t)\,\frac{dL}{dt}\,dt = \frac{1}{2}\,E\,A\,\frac{(L_1 - L_0)^2}{L_0}. \qquad \text{(fo.12)}$$

Note that this formula depends only on L_1 and L_0, not how we go from one to the other. Think now of the whole string as composed of a sequence of infinitesimal straight segments. Then we can use (fo.12) to write

$$\mathcal{V}_{\text{str}} = \frac{1}{2}\,E\,A\int\left(\sqrt{\dot{X}^2 + \dot{Y}^2} - 1\right)^2 ds, \qquad \text{(fo.13)}$$

where points on the string is labeled by s (the arclength along the string at equilibrium), the string is given by $x = X(s)$ and $y = Y(s)$, and the dots indicate derivatives with respect to s.

Detail: Take, in (fo.12), $L_0 = ds$. Then $L_1 = \sqrt{\dot{X}^2 + \dot{Y}^2}\,ds$, and $\mathcal{V}_{\text{seg}} = \frac{1}{2}\,E\,A\left(\sqrt{\dot{X}^2 + \dot{Y}^2} - 1\right)^2 ds$.

2.2 Answer: Small transversal vibrations of a beam

With the approximations used, the transversal momentum density (momentum per unit length) is $\rho\, u_t$. The momentum flux is given by the shear force f_s in (fo.8). Conservation yields

$$(\rho\, u_t)_t + (E\, I\, u_{xxx})_x = 0. \quad \text{Homogeneous beam equation.} \tag{2.1}$$

Conservation of energy. Equation (fo.6) yields the potential (elastic) energy. The energy density is $e_d = \frac{1}{2}\,\rho\, u_t^2 + \frac{1}{2}\, E\, I\, u_{xx}^2$. The energy flux is provided by the work per unit time done by the shear force, $f_s\, u_t$, and the work done per unit time by the torque, $\tau_b\, u_{xt}$ (note that $u_x = \tan\theta = \theta$ in this linear approximation, thus $\theta_t = u_{xt}$). Putting this all together gives the *conservation of energy* equation

$$\left(\frac{1}{2}\,\rho\, u_t^2 + \frac{1}{2}\, E\, I\, u_{xx}^2\right)_t + \Big(E\, I\, u_{xxx}\, u_t - E\, I\, u_{xx}\, u_{xt}\Big)_x = 0. \tag{2.2}$$

By direct differentiation it is easy to see that the solutions to (2.1) satisfy (2.2).

One may ask the question: *why is it that conservation of energy does not yield a new equation?* The answer (or, at least, one answer) is that, in the situation being considered there is no mechanism for energy transfer between mechanical and internal energy. There are no losses of mechanical energy. Of course, in the real world a beam does not vibrate for ever (unless energy is continuously supplied to it). The vibrations are damped, and eventually become heat.

Conservation of angular momentum. If we use the origin as the center for the angular momentum, the angular momentum density is $x\,\rho\, u_t$. The angular momentum flux is given by τ_b, and the torque produced by the shear force: $x\, f_s$. The conservation of angular momentum gives the equation

$$(x\,\rho\, u_t)_t + (E\, I\, x\, u_{xxx} - E\, I\, u_{xx})_x = 0. \tag{2.3}$$

It is easy to see that this is the same as (2.1).

Note. *Let $\vec{r} = (X,\, Y,\, z)$ be the position vector for a mass-element in the beam — as given by (fo.4) for each fixed $(x,\, y,\, z)$. Then the angular momentum of the element is $m_{op} = \mu\, \vec{r}\times\vec{v}\, dx\, dy\, dz$, where $\vec{v} = \frac{d}{dt}\vec{r} = (-y\, u_{xt},\, u_t,\, 0)$ is the parcel velocity and μ is the mass-density (mass per unit volume) of the beam material — assumed constant. To be consistent with prior approximations, we should* **neglect the longitudinal component of the velocity,** *and write $\vec{v} \approx (0,\, u_t,\, 0)$. Thus $m_{op} \approx \mu\,(-z\, u_t,\, 0,\, X\, u_t)\, dx\, dy\, dz$. Upon neglecting the nonlinear terms as well, this yields $m_{op} \approx \mu\,(-z\, u_t,\, 0,\, x\, u_t)\, dx\, dy\, dz$. Finally, integrating over the beam cross-section — $|z| \le \frac{1}{2}\, w$ and $|y| \le \frac{1}{2}\, h$ — we obtain the angular momentum density $(0,\, 0,\, \rho\, x\, u_t)$. This has a single non-zero component, whose conservation yields (2.3).*

Boundary conditions. Using (fo.10) we see that:

At a clamped end both u and u_x are prescribed — e.g., $\boldsymbol{u = u_x = 0}$.

At a free end the torque and the shear should vanish, thus $\boldsymbol{u_{xx} = u_{xxx} = 0}$.

At a hinged end there should be no torque, but position is given — e.g., $\boldsymbol{u = u_{xx} = 0}$.

A situation where $u_x = u_{xxx} = 0$ at an end could be devised as follows: have a vise that can freely (no friction) slide up and down a vertical rod at, say, $x = 0$. Then clamp the blade to the vise. In this case there is no shear at the end, so that $u_{xxx} = 0$, and $u_x = 0$ because of the clamping. But neither the torque, nor the position of the beam are prescribed there.

Challenge questions:

Is it possible to design a situation such that $u = u_{xxx} = 0$ at an end? How about $u_x = u_{xx} = 0$?

3 Kelvin waves in a rotating basin

3.1 Statement: Kelvin waves in a rotating basin

Waves trapped near a boundary can occur via a wave-guide effect, when the waves velocity is lower near the boundary than elsewhere; e.g.: edge waves. Rotation can do the same. The simplest example exhibiting this behavior are the (linear) shallow water waves with constant rotation (f-plane approximation)

$$\boldsymbol{\eta_t + h\,(u_x + v_y) = 0,} \tag{3.1}$$

$$\boldsymbol{u_t + g\,\eta_x = f\,v,} \tag{3.2}$$

$$\boldsymbol{v_t + g\,\eta_y = -f\,u,} \tag{3.3}$$

where η is the surface deviation from equilibrium, $h > 0$ is the (constant) depth, u and v are the two components of the flow velocity, g is the acceleration of gravity, and $f \neq 0$ is the (constant) rotation angular velocity.
Consider these equations in the region $\boldsymbol{y > 0}$, with **boundary condition $\boldsymbol{v = 0}$ at $\boldsymbol{y = 0}$,** and **search for (non-trivial) solutions that**
(i) Are functions of $(x - s\,t)$, and y only, where s is some constant.
(ii) Are exponentially trapped near the $y = 0$ boundary.

Hint. *Look for solutions of the form* $\eta = N(x - s\,t)\,e^{-\lambda\,y}$, $u = U(x - s\,t)\,e^{-\lambda\,y}$, *etc., where* s *and* λ *are constants such that* $\mathrm{Re}(\lambda) > 0$. Eliminate from consideration solutions where N and U are constants — that is: "trivial" solutions.

3.2 Answer: Kelvin waves in a rotating basin

Notice that, if $v = V(x - s\,t)\,e^{-\lambda\,y}$, then (since $v = 0$ at $y = 0$) it **must be $\boldsymbol{v \equiv 0}$.** Hence the equations reduce to

$$-s\,N' + h\,U' = 0, \quad -s\,U' + g\,N' = 0, \quad \text{and} \quad g\,\lambda\,N = f\,U. \tag{3.4}$$

From the first two equations it follows that $-s\,N + h\,U = \alpha$ and $g\,N - s\,U = \beta$, where α and β are constants. For a non-trivial solution — i.e.: we do not want N and U both constant — the determinant of this linear system for N and U must vanish. This gives

$$s^2 = g\,h \quad \text{and} \quad U = \gamma + \frac{s}{h}\,N, \tag{3.5}$$

where $\gamma = \alpha/h$ and $\beta = -s\,\gamma$. Then, from the equation $g\,\lambda\,N = f\,U$, we conclude that $\gamma = 0$ and $f\,s = g\,h\,\lambda = s^2\,\lambda$. But λ must have positive real part. Hence

$$\boldsymbol{s = \sigma\,\sqrt{g\,h}} \quad \text{and} \quad \boldsymbol{\lambda = \frac{|f|}{\sqrt{g\,h}},} \tag{3.6}$$

where $\boldsymbol{\sigma = \mathrm{sign}(f)}$. The solutions are then

$$\boldsymbol{\eta = N(x - s\,t)\,e^{-\lambda\,y}, \quad u = \frac{s}{h}\,N(x - s\,t)\,e^{-\lambda\,y},} \quad \text{and} \quad \boldsymbol{v = 0,} \tag{3.7}$$

where $\boldsymbol{N}$ is an arbitrary function.

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4 Amplitude modulation with frozen carrier frequency

4.1 Statement: Amplitude modulation with frozen carrier frequency

Consider a dispersive equation in n-D of the form $$\boldsymbol{u_t + i\,\Omega(-i\,\nabla)\,u = 0,} \tag{4.1}$$
where: (i) $u = u(\vec{x}, t)$ is a scalar function, (ii) ∇ is the gradient operator in n-D, and (iii) $\omega = \Omega(\vec{k})$ is a smooth, real valued (scalar), of the wave-number vector $\vec{k} \in \mathcal{R}^n$. †

† If you want to, you may assume that Ω is a polynomial: a finite sum $\boldsymbol{\Omega = \sum a_{i_1 i_2 \dots i_n} k_1^{i_1} k_2^{i_2} \dots k_n^{i_n}}$, for some coefficients $\boldsymbol{a_{i_1 i_2 \dots i_n}}$. However, this is not actually necessary.

This equation is dispersive, with dispersion relation $\omega = \Omega(\vec{k})$.
Consider now solutions of the form

$$\boldsymbol{u \sim A(\vec{\chi},\, \tau)\, e^{i\,\theta}}, \tag{4.2}$$

with $\boldsymbol{\vec{\chi} = \epsilon\, \vec{x}}$, $\boldsymbol{\tau = \epsilon\, t}$, and $\boldsymbol{\theta = \vec{k}_0 \cdot \vec{x} - \omega_0\, t}$, where: (i) $0 < \epsilon \ll 1$ is a constant, (ii) $\vec{k}_0$ is a constant wave-number vector, and (iii) $\omega_0 = \Omega(\vec{k}_0)$.

Derive the leading order equation satisfied by $\boldsymbol{A}$.

Hint. *Think of u as a function of $\vec{x}$ and $\vec{\chi}$, with $\vec{x}$ and $\vec{\chi}$ independent variables (same for t and τ). Then the equation can be written in the form*

$$\boldsymbol{u_t + \epsilon\, u_\tau + i\, \Omega(-i\, \nabla_1 - i\, \epsilon\, \nabla_2)\, u = 0}, \tag{4.3}$$

where ∇_1 is the gradient operator for $\vec{x}$ and ∇_2 is the gradient operator for $\vec{\chi}$. Now expand the operator $\Omega(-i\,\nabla_1 - i\,\epsilon\,\nabla_2)$ in powers of ϵ, and notice that evaluating functions of $-i\,\nabla_1$ is easy — just replace $-i\,\nabla_1$ by $\vec{k}_0$ (do you see why this?).

Remark 4.1 Consider a function, $f = f(x)$, evaluated on an operator (or matrix) $f = f(A)$.† When doing this we need to keep in mind that, because operators do not commute, the "regular" rules of calculus may not apply. For example: (i) Generally we can write $\frac{\mathrm{d}}{\mathrm{d}t} f(A(t)) = f'(A) \frac{\mathrm{d}A}{\mathrm{d}t}$ only if A and $\frac{\mathrm{d}A}{\mathrm{d}t}$ commute. (ii) Generally we can "Taylor" expand $f(A+B)$, for B small, only if A and B commute. ♣

† This can certainly be done when $\boldsymbol{f}$ is a polynomial. But this is not the only case: For example, if $\boldsymbol{f}$ is analytic, we can use Cauchy's theorem to write $\boldsymbol{f(A) = \frac{1}{2\pi i} \oint (z - A)^{-1} f(z) \mathrm{d}z}$, where the contour of integration includes the spectrum of A.

4.2 Answer: Amplitude modulation with frozen carrier frequency

We follow the hint, and write†

$$\boldsymbol{i\, \Omega(-i\, \nabla_1 - i\, \epsilon\, \nabla_2) = i\, \Omega(-i\, \nabla_1) + \epsilon\, \vec{c}_g(-i\, \nabla_1) \cdot \nabla_2 + O(\epsilon^2)}. \tag{4.4}$$

Using this in (4.3), with u as in (4.2), yields

$$\left(-i\, \omega_0\, A + \epsilon\, A_\tau + i\, \Omega(\vec{k}_0)\, A + \epsilon\, \vec{c}_g(\vec{k}_0) \cdot \nabla_2\, A + O(\epsilon^2)\right)\, e^{i\,\theta} = 0. \tag{4.5}$$

Since $\omega_0 = \Omega(\vec{k}_0)$, we obtain (at leading order)

$$\boldsymbol{A_\tau + \vec{c}_g(\vec{k}_0) \cdot \nabla_2\, A = 0}. \tag{4.6}$$

† We can do this because ∇_1 and ∇_2 commute. See remark **4.1**.

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5 Two strings connected by a third one

5.1 Statement: Two strings connected by a third one

Consider two semi-infinite strings under tension, connected by a short piece of another string. If $u = u(x,\, t)$ denotes the transverse deviations of the strings, in the linear (small deviations and small slopes), the equation describing the system is

$$0 = u_{tt} - c^2\, u_{xx} \tag{5.1}$$

where $\boldsymbol{c = c_1}$ for $\boldsymbol{x < 0}$, $\boldsymbol{c = c_2}$ for $\boldsymbol{0 < x < L}$, $\boldsymbol{c = c_3}$ for $\boldsymbol{L < x}$, the $\boldsymbol{c_j > 0}$ are constants,[9] and $\boldsymbol{L}$ is the length of the connecting piece. This equation must be supplemented by the conditions

$$u \quad \text{and} \quad u_x \quad \text{are continuous across the joints at } \; x = 0 \; \text{ and } \; x = L. \tag{5.2}$$

[9] Given by $\boldsymbol{c_j = \sqrt{T/\rho_j}}$, where $\boldsymbol{T}$ is the (common) string tension and $\boldsymbol{\rho_j}$ are the respective string densities. As usual, we assume a tension that is constant throughout, so that neglecting longitudinal movements is justified.

Assume now that a monochromatic wave of wave-frequency ω arrives at the (composite) junction. Then:

1. **Calculate the reflection and transmission coefficients across the (composite) junction.** That is, find solutions to the equation of the following form:

$$\boldsymbol{u} = \begin{cases} \boldsymbol{e^{i\,\omega\,(t-x/c_1)}} + \boldsymbol{R\,e^{i\,\omega\,(t+x/c_1)}} & \text{for } \boldsymbol{x < 0} \\ \boldsymbol{a\,e^{i\,\omega\,(t-x/c_2)}} + \boldsymbol{b\,e^{i\,\omega\,(t+x/c_2)}} & \text{for } \boldsymbol{0 < x < L} \\ \boldsymbol{\mathcal{T}_c\,e^{i\,\omega\,(t-x/c_3)}} & \text{for } \boldsymbol{L < x} \end{cases} \tag{5.3}$$

   where $\boldsymbol{\omega > 0}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{R}$, and $\boldsymbol{\mathcal{T}_c}$ are constants. Note: ω is a free constant, everything else is a function of ω. Important: Make sure that your calculations do not involve division by zero!

2. **Why do all the terms in (5.3) have the same time dependence? How does this follow from (5.2)?**

3. **Under which conditions is there no reflection, $R = 0$ (i.e.: impedance matching)?**

4. **Is it possible to get no transmission, $\mathcal{T}_c = 0$?**

This problem illustrates the principle behind lens coating, used in optics to prevent reflection losses in complex multi-lense devises, — *in a very simplified context.*

5.2 Answer: Two strings connected by a third one

Part 2

The equations in (5.2) must hold for all times, at $x = 0$ and at $x = L$. Hence, when the waves have an exponential dependence in time (as in this problem), (5.2) can apply only if all the exponentials are the same. Hence the use of the same time dependence for all the terms in (5.3).

Physically: if the piece of string immediately to the left of some point x_* vibrates with frequency ω, the piece immediately to the right of x_* (which is attached to the piece on the left) must vibrate with the same frequency. Hence, the whole string must vibrate with the same frequency. *These arguments apply to a string in "steady" state only,* else we cannot even talk about "frequency".

Part 1

Substituting (5.3) into (5.2) yields the following set of equations

$$1 + R = a + b \quad \text{and} \quad 1 - R = \frac{c_1}{c_2}\,(a - b) \tag{5.4}$$

from the $x = 0$ equation, and

$$A + B = D \quad \text{and} \quad A - B = \frac{c_2}{c_3}\,D \quad \Longleftrightarrow \quad A = \frac{c_3 + c_2}{2\,c_3}\,D \quad \text{and} \quad B = \frac{c_3 - c_2}{2\,c_3}\,D \tag{5.5}$$

from the $x = L$ equation, where

$$\boldsymbol{A = a\,e^{-i\,\phi},\ B = a\,e^{i\,\phi},\ D = \mathcal{T}_c\,e^{-i\,\psi},\ \phi = \omega\,L/c_2,} \text{ and } \boldsymbol{\psi = \omega\,L/c_3.} \tag{5.6}$$

Hence

$$a = \frac{c_3 + c_2}{2\,c_3}\,D\,e^{i\phi} \quad \text{and} \quad b = \frac{c_3 - c_2}{2\,c_3}\,D\,e^{-i\,\phi}. \tag{5.7}$$

But (5.4) is equivalent to

$$1 = \frac{c_2 + c_1}{2\,c_2}\,a + \frac{c_2 - c_1}{2\,c_2}\,b \quad \text{and} \quad R = \frac{c_2 - c_1}{2\,c_2}\,a + \frac{c_2 + c_1}{2\,c_2}\,b. \tag{5.8}$$

The first of these two equations then yields

$$e^{i\psi} = \left(\frac{c_2 + c_1}{2\,c_2}\,\frac{c_3 + c_2}{2\,c_3}\,e^{i\phi} + \frac{c_2 - c_1}{2\,c_2}\,\frac{c_3 - c_2}{2\,c_3}\,e^{-i\phi} \right) \mathcal{T}_c, \tag{5.9}$$

which **determines the transmission coefficient** $\mathcal{T}_c$. Then, from the second equation

$$R = \left(\frac{c_2 - c_1}{2\,c_2}\,\frac{c_3 + c_2}{2\,c_3}\,e^{i\phi} + \frac{c_2 + c_1}{2\,c_2}\,\frac{c_3 - c_2}{2\,c_3}\,e^{-i\phi} \right) \mathcal{T}_c\,e^{-i\psi}, \tag{5.10}$$

which **gives the reflection coefficient** R.

Remark 5.1 *Note that* $(c_2 + c_1)(c_3 + c_2) > |(c_2 - c_1)(c_3 - c_2)|$, *since all the* c_j*'s are positive. Thus, the coefficient in front of* $\mathcal{T}_c$ *on the right in (5.9)* **never vanishes.**
Hence **the transmission coefficient is always defined. In fact** .... $0 < |\mathcal{T}_c| < \infty$.

Alternatively, introduce

$$R_{ij} = \frac{c_i - c_j}{c_i + c_j} \quad \text{and} \quad \gamma = \frac{4\,c_2\,c_3}{(c_1 + c_2)(c_2 + c_3)} = (1 - R_{12})(1 - R_{23}), \tag{5.11}$$

where we notice that $-1 < R_{ij} < 1$. Then the equations above take the form

$$e^{i\psi} = \frac{e^{i\phi}}{\gamma}\left(1 + R_{12}\,R_{23}\,e^{-i2\phi}\right)\mathcal{T}_c \quad \text{and} \quad R = -\frac{e^{i\phi}}{\gamma}\left(R_{12} + R_{23}\,e^{-i2\phi}\right)\mathcal{T}_c\,e^{-i\psi}. \tag{5.12}$$

Hence

$$\mathcal{T}_c = \frac{\gamma\,e^{i(\psi-\phi)}}{1 + R_{12}\,R_{23}\,e^{-i2\phi}} \quad \text{and} \quad R = -\frac{R_{12} + R_{23}\,e^{-i2\phi}}{1 + R_{12}\,R_{23}\,e^{-i2\phi}}, \tag{5.13}$$

where we note that the coefficients depend on ωL via ψ and ϕ.

Part 3

No reflection, $R = 0$. This can happen only for $R_{12} = \pm R_{23}$, which leads to the following cases

Case $c_3 = c_1$. Then $R_{12} = -R_{23}$.
The **no-reflection condition** is $\phi = \frac{\omega L}{c_2} = n\,\pi$, where n is an integer. That is:
A wave in the connecting segment travels back and forth (distance $2L$) in n wave periods.
Not an interesting case, since no connection segment is needed at all to suppress reflection!

Case $c_2^2 = c_1\,c_3$. Then $R_{12} = R_{23}$.
The **no-reflection condition** is $\phi = \frac{\omega L}{c_2} = \left(n + \frac{1}{2}\right)\pi$, where n is an integer. That is:
A wave in the connecting segment travels back and forth (distance $2L$) in $n + \frac{1}{2}$ wave periods. This situation corresponds to the "simplest" explanation of how come there is no reflection. The idea is that, because of the 1/2 period fraction in the travel time, the waves reflected from the first interface (at $x = 0$), and those from the second (at $x = L$), have a π phase difference, and hence cancel each other. This is, basically, correct, but it is also an over-simplification. It predicts no reflection for any value of c_2, which (as shown here) is not true.

Part 4

From remark **5.1**, it follows that **there is always transmission. Total reflection is not possible.**

6 Optical ray in a stratified media

6.1 Statement: Optical ray in a stratified media

Consider a **nearly straight and nearly horizontal light ray in a slightly stratified media.** The equation for the ray is

$$\frac{d}{d\tau}\left(\frac{n}{v}\frac{d\vec{x}}{d\tau}\right) = v\,\nabla n, \quad (6.1)$$

where τ is some arbitrary parameter, c_0 is the speed of light in vacuum, $n = c_0/c$ is the index of refraction, and $v = |\frac{d\vec{x}}{d\tau}|$. You can find this equation in the *Lectures on Average Lagrangian*, section: *Examples*, subsection: *Wave equation and the Eikonal equation*, subsubsection: *Fermat's principle*.

Here we will assume 2-D, with $\vec{x} = (x,\, y)$ (x being the horizontal coordinate and y the vertical), with the weak stratification characterized by $n = n_0\,(1 - \epsilon\,(y - y_0))$, with $n_0 > 0$ and $0 < |\epsilon|\,L \ll 1$ (L is a length scale). *You have now the following tasks:*

Task #1. Approximately compute the ray defined by the properties: it is horizontal at $x = 0$, with $y = y_0$.

Task #2. What value of ϵ yields $y = y_0 - 10$ m for $x = 10$ km?

For task #1. Use x as the parameter for the ray, i.e.: $y = y(x)$, and **show that** the two equations in (6.1) reduce to the single one

$$\frac{d}{dx}\frac{n}{v} = 0. \quad (6.2)$$

To do this, first **show that $y' = \frac{dy}{dx}$ cannot vanish** — except for isolated points; e.g.: $x = 0$. That is: **show that** *solutions to* (6.2) *that vanish on an interval do not correspond to solutions to* (6.1). As a side issue, answer the question:

Task #3. Does (6.2) have solutions such that y' vanishes of an interval, but not everywhere?

Task #4. The approximate solution that you will obtain using hint #1 below is not be valid for all values of x, because it does not satisfy "y' is small" everywhere. **Where is this solution valid?**
Important: when doing task #2, **verify that you use the approximate solution within its range of validity.**

Task #5. Finally, describe real world phenomena that this calculation can help to elucidate.

Hint #1. Use that $y \approx y_0$ to solve (6.2). In particular, that $\frac{dy}{dx} = y'$ **is small.**

Hint #2. For task #3, look at what the equation reduces to when y' is small (i.e.: use the result of hint #1).

6.2 Answer: Optical ray in a stratified media

For $y = y(x)$, $v = \sqrt{1 + (y')^2}$. Then (6.1) reduces to: (6.2) and

$$\frac{d}{dx}\left(\frac{n}{v}\,y'\right) = v\,n_y. \quad (6.3)$$

Note: (6.2) follows because $n_x = 0$.
It is easy to see that **this equation does not allow $y' = 0$ on an interval,** because then (in the interval) the left hand side would vanish, while the right hand side would not.
Next we note that (6.2) can be integrated to

$$n(y) = \mu\,v = \mu\,\sqrt{1 + (y')^2}, \quad (6.4)$$

where μ is a constant. Differentiating this yields $n_y\,y' = (\mu/v)\,y'\,y''$, which (since $y' \neq 0$) leads to

$$v\,n_y = \mu\,y''. \quad (6.5)$$

However (6.3) is equivalent to

$$0 = \frac{d}{dx}\left(\frac{n}{v}\right)y' + \left(\frac{n}{v}\,y'' - v\,n_y\right) = \frac{d}{dx}\left(\frac{n}{v}\right)y' + (\mu\,y'' - v\,n_y), \quad (6.6)$$

where we used $n/v = \mu$. From (6.2) and (6.5), (6.6) is satisfied. **We conclude that (6.2) implies (6.3), as long as we avoid solutions with $y' \equiv 0$.**
Hence **we only need to solve** (6.2) — equivalently (6.4).
Next we use hint #1, approximate $v =\approx 1 + \frac{1}{2}\,(y')^2$, and substitute in (6.4), to get

$$\frac{n_0}{\mu}\,(1 - \epsilon\,(y - y_0) \approx 1 + \frac{1}{2}\,(y')^2. \quad (6.7)$$

It follows that $\mu \approx n_0$ and

$$(y')^2 \approx -2\,\epsilon\,(y - y_0). \quad (6.8)$$

See **note #1** below.

Given the condition at $x = 0$ in **task #1**, we conclude that

$$y \approx y_0 - \tfrac{1}{2}\,\epsilon\,x^2. \quad (6.9)$$

See **note #2** below.

Answer to **task #2.** Clearly $$\epsilon = 2 \times 10^{-7}\text{m}^{-1}. \quad (6.10)$$
Note that the condition in **Note #2** is valid for $x = 10\,$km.

Note #1. Equation (6.8) has solutions that satisfy $y' = 0$ in an interval. Specifically: $y = y_0$ for $x_1 \leq x \leq x_2$, $y = y_0 - \frac{1}{2}\,\epsilon\,(x - x_1)^2$ for $x \leq x_1$, and $y = y_0 - \frac{1}{2}\,\epsilon\,(x - x_2)^2$ for $x \geq x_2$ — where $x_1 \leq x_2$ are arbitrary. However, as shown earlier, only the case $x_1 = x_2$ is consistent with (6.3). The **same result applies to (6.2 / 6.4),** since (6.8) describes the behavior of (6.2) when y' is small. This is **task #3**.

Note #2. (6.9) breaks the y' small assumption for large x, and **it is valid for $|x| \ll |\epsilon|^{-1}$ only.** This is **task #4**.

Finally, **discussion; task #5.** The index of refraction of air, at sea level, is $n = 1.0003 + O(10^{-5})$, where the "uncertainty" is due to temperature, humidity, and pressure dependence. Hence a stratification of the size displayed in (6.10), or even much larger, is not only possible, but likely. Of course, just as possible are horizontal variations in n, which bend the ray's laterally. These are examples of phenomena caused by these variations:

1. **Mirrors on the road.** While driving in hot, dry, weather, you may see the road "wet" far ahead of you. This is caused because the hot asphalt in the road causes a temperature stratification, which then causes n to decrease with height ($\epsilon < 0$), so light rays are bent upwards, "reflecting" from the road.
2. **Seeing things over the horizon,** particularly over a cold body of water (or, at least, cooler than the atmosphere above it). This is the reverse of item **1**.
3. **Star position shifting.** Bending causes the apparent direction the rays are coming from to change, leading to the star seeming to shift position in the sky.
4. **Twinkling of stars.** This is also attributed to ray paths being perturbed by variations in n. But I do not buy the explanations in the "popular" science literature.[10] So I am not going to say much here, other than I think that the difference between items **3** and **4** is one of scale: item **3** requires all the rays arriving to your eye (or instrument used to observe) being deviated in the same way, while item **4** happens when the rays have varying directions on the scale of the eye.

A 2000+ year old conspiracy, spanning from the Greeks to NASA.

The *ancient Greeks already knew that the Earth was round,* and managed to *calculate its radius with less than a 1% error.* This was done by Eratosthenes around 200 B.C. He knew that at noon on the first day of summer, the Sun passed directly overhead at Syene, Egypt (current Aswan). At midday of the same day, he measured the angular displacement of the Sun from overhead at Alexandria, Egypt (7.2 degrees). Knowing the distance from Syene to Alexandria, he could then compute the radius of the Earth. In current units the number he obtained was 6366 km — compare this with the 6378 km modern value obtained from satellite measurements, astoundingly accurate!

In 1838, Samuel Rowbotham [alias Paralax], a medical doctor, decided to directly verify the Earth curvature. For this he selected a straight stretch of a canal (about 6 miles long) in Cambridgeshire, UK. He then secured a small boat with a 3 ft high mast, and went into the water with a small telescope which he held above the water level at about 8 in. The boat was then moved away from him (by someone else, I assume), as he watched it with the telescope. *He observed that the mast was still visible at a distance of 6 miles.* Prior to this he had computed that, at that distance, the curvature of the Earth should have dropped the boat well below his "straight line of vision" — by several feet, see (6.11). Hence, *assuming that light moves in a straight line, he concluded that the Earth was flat.*[11] Unfortunately he collected no data on water/air temperature, humidity, pressure, etc., from which to estimate what

[10] Many include pictures showing one ray going from the star to the eye, forgetting that the star illuminates the whole Earth (and more). One ray randomly kicked out of your pupil will be replaced by a nearby one; so why does this change apparent luminosity? This is too imprecise.

[11] **The parameters in task #2 are inspired by this experiment; with 10 km ≈ 6 miles. See note #3.**

the index of refraction was along the 6 miles of the channel. He did not seem to be aware that this was an issue.

Rowbotham describes his experiment, as well as other observations (all ignoring refraction), in a book (reference below) — you can read there what "zetetic" means. This was the beginning of the modern "Flat Earth" phenomena.

Note #3. Let ℓ **be the distance between two points,** P_1 and P_2, on the Earth surface. Consider the tangent line to the surface at P_1, in the plane determined by the two points and the center of the Earth, C. Let d **be the distance from this line to** P_2. **We want to determine d as a function of ℓ.**

Let θ be the angle between the straight segments $\overline{CP_1}$ and $\overline{CP_2}$. If θ is small,[12] we can make the following approximations: $d = r\,(1-\cos\theta) \approx \frac{1}{2}\,r\,\theta^2$, and $\theta \approx \frac{\ell}{r}$, where $r \approx 6378$ **km is the radius of the Earth.** Hence

$$d \approx \frac{\ell^2}{2\,r}, \tag{6.11}$$

which is **valid for $\ell \ll r$. For $\ell = 10$ km, this yields $d \approx 7.84$ m.**

Reference. *Zetetic astronomy: earth not a globe!* by Paralax.
A description of Several Experiments which prove that the surface of the sea is a perfect plane and that the Earth is not a globe. (Dec. 8, 1848) Paper read before the Royal Astronomical Society.
You can find the book at:
https://ia802705.us.archive.org/30/items/zeteticastronom00rowbgoog/zeteticastronom00rowbgoog.pdf

THE END.

[12] Note that, for $|\theta| \ll 1$, it does not matter if we measure ℓ along the surface, or along the straight segment $\overline{P_1P_2}$.