18.376 Problem Set #02, MIT Spring 2023

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Due: Last day of lectures, Spring 2023.

Turn it in pdf format via the Canvas web page.

Note. Answers to all the problems will be posted, but *only some problems will be graded.*
The graded problems are "quiz #02".

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1 Edge waves for a pice-wise constant wave equation

Statement: Edge waves for a pice-wise constant wave equation

Consider the wave equation

$$u_{tt} - c^2\,(u_{xx} + u_{yy})) = 0, \quad \text{for } \; y > 0, \tag{1.1}$$

where $c = c_2$ for $y > L$, $c = c_1$ for $0 < y < L$, $c_2 > c_1 > 0$ are constants, and $L > 0$ is a constant. Furthermore, **u and u_y are continuous across the $y = L$ interface, and the boundary condition $u = 0$ applies at $y = 0$.**

Find all the edge waves for this problem. Namely, non-vanishing solutions to the problem above of the form

$$u = U(y)\,\exp\left(i\,(k\,x - \omega\,t)\right), \tag{1.2}$$

where $k > 0$ and ω are real constants, and U vanishes (exponentially) as $y \to \infty$.

Hint. For any given fixed k, the problem will lead to an eigenvalue problem, with eigenvalue ω^2, and eigenfunction $U = U(y)$. This problem can be then reduced to a transcendental equation that ω^2 must satisfy. In order to study the solutions to this later equation, it may be useful to write it down in terms of the variable $\Delta = (L/c_1)\,\sqrt{\omega^2 - k^2\,c_1^2}$, which is restricted to the range

$$0 < \Delta < \Delta_M = k\,(L/c_1)\,\sqrt{c_2^2 - c_1^2}, \tag{1.3}$$

because it must be that $k^2\,c_1^2 < \omega^2 < k^2\,c_2^2$. By the way: **you MUST show that this restriction on ω^2 is needed.** You cannot use this just because I said so here.

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2 Elastic Hanging String

Statement: Elastic hanging string

Consider an elastic string, with **constant mass per unit length ρ**, and **constant cross-sectional area A**. Assume small deformations, so that the elastic forces (tension) generated on any small portion of the string have the form

$$T = E\,A\,\frac{\Delta\ell}{\ell} = \kappa\,\frac{\Delta\ell}{\ell} \tag{2.1}$$

where ℓ is the un-stretched length of the string segment, $\Delta\ell$ is the length change, E is the Young's modulus for the string material, and $\kappa = E\,A$. Notice that we ignore any changes in A that may occur because of the stretching.

Assume now that the string is hanging vertically, and straight — no lateral displacements, from some fixed point, so that **it can be described by a function $Z = Z(\zeta,\, t)$,** where

- **A.** z is the vertical coordinate, $z = 0$ is the position of the point to which the string is attached, and the relaxed length of the string is L
- **B.** $z = \zeta$, $-L \le \zeta \le 0$, would be the vertical position of a mass element along the string when not stretched. Thus ζ serves as a label for the mass elements of the string.
- **C.** $z = Z(\zeta,\, t)$ is the actual position of the mass element whose label is ζ. Hence the displacement field along the string is given by $u = u(\zeta,\, t) = Z - \zeta$.

Perform the following tasks:

1. Derive an equation for u, assuming that the only external force on the string is gravity, characterized by g.

2. What are the boundary conditions for the equation derived in **1**?
3. How do the boundary conditions change if there is a mass $\boldsymbol{m}$ attached to the lower end of the string?
4. What is the equilibrium (no motion) state for the string, as described by the equation and boundary conditions in **2** and **3**? Call this solution $\boldsymbol{u_* = u_*(\zeta)}$.
5. The fundamental modes of vibration for the system in this problem are described by solutions of the form $\boldsymbol{u = u_*(\zeta) + a\ \cos(\omega\, t)\ \sin(k\, \zeta)}$, where (obviously) $a,\ k \neq 0$. **Find equations for $\boldsymbol{\omega}$ and $\boldsymbol{k}$.** *Why is the ζ dependence via a sine?*

3 Gravity water waves (dispersion relation)

Statement: Gravity water waves (dispersion relation)

The equations for (infinitesimal) irrotational surface waves on a liquid over a flat impermeable bottom, when surface tension and dissipative effects are neglected, are

$$\begin{aligned}
\Delta\Phi &= 0, \quad \text{for } 0 < z < h \quad \text{(incompressibility).} && (3.1)\\
\Phi_z &= 0, \quad \text{for } z = 0 \quad \text{(impermeable bottom).} && (3.2)\\
\eta_t - \Phi_z &= 0, \quad \text{for } z = h \quad \text{(kinematic boundary condition).} && (3.3)\\
\Phi_t + g\,\eta &= 0, \quad \text{for } z = h \quad \text{(dynamic boundary condition).} && (3.4)
\end{aligned}$$

Here (i) $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$ is the Laplace operator, (ii) $\Phi = \Phi(x,\, y,\, z,\, t)$ is the velocity potential — the flow velocity is given by $\vec{u} = \text{grad}\,\Phi$, (iii) $\vec{x} = (x,\, y)$ are the horizontal coordinates, (iv) z is the vertical coordinate — $z = 0$ is the bottom and $z = h$ is the equilibrium level for the liquid (h is a constant), (v) η is the deviation from equilibrium of the surface — the surface is at $z = h + \eta(x,\, y,\, t)$, and (vi) g is the acceleration of gravity. Equation (3.2) is the statement that there is no flow through the bottom, equation (3.3) states that the velocity of the surface normal to itself is equal to the flow velocity normal to the surface, and equation (3.4) follows from the balance of forces at the interface — Bernoulli's principle.

Compute the dispersion relation for these equations: separate the time and horizontal dependence as $\boldsymbol{e^{i\,\theta}}$ — where $\boldsymbol{\theta = \vec{k}\cdot\vec{x} - \omega\, t}$ and $\boldsymbol{\vec{x} = (x,\ y)}$, solve for the vertical dependence, and find the equation that relates $\boldsymbol{\omega}$ and $\boldsymbol{\vec{k}}$.

4 Narrow band wave packages #01

4.1 Statement: Narrow band wave packages #01

Consider a solution to a 1-D linear dispersive system with a narrow band spectrum and a single branch of the dispersion relation active.[1] Using the Fourier Transform, this means that the solution can be written in terms of a scalar function of the form

$$u = u(x,\, t) = \int_{-\infty}^{\infty} U(k)\, e^{i\,(k\,x - \omega\,t)}\, dk, \qquad \text{where:} \tag{4.1}$$

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[1] If more than one exists.

a. The wave-frequency $\omega = \omega(k)$ is given by the dispersion relation. Assume that ω is a smooth, real valued, function.

b. The complex amplitude U is concentrated near some wave-number k_0.

That is[2]
$$U = \frac{1}{\epsilon}\, A\left(\frac{k-k_0}{\epsilon}\right), \tag{4.2}$$
where $0 < \epsilon \ll 1$ and A is a smooth function that decays rapidly at infinity.
We use a-dimensional variables, otherwise a statement like $0 < \epsilon \ll 1$ *has no meaning.*

Assume that $\boldsymbol{\frac{d^2\omega}{dk^2}(k_0) = 0}$ and $\boldsymbol{\mu_0 = \frac{d^3\omega}{dk^3}(k_0) \neq 0}$. Thus the Taylor expansion for ω centered at k_0 has the form

$$\omega(k) = \omega_0 + c_0\,(k-k_0) + \frac{1}{6}\,\mu_0\,(k-k_0)^3 + \ldots, \tag{4.3}$$

where $\omega_0 = \omega(k_0)$ and $c_0 = c_g(k_0) = \frac{d\omega}{dk}(k_0)$ is the group speed at k_0. Then **show that** u in (4.1) has the form of a modulated carrier wave

$$\boldsymbol{u = a(X,\, T)\, e^{i\,(k_0\,x - \omega_0\,t)}}, \tag{4.4}$$

where $\boldsymbol{X = \epsilon\, x}$, $\boldsymbol{T = \epsilon\, t}$, and the modulation amplitude satisfies the equation

$$\boldsymbol{a_T + c_0\, a_X - \frac{1}{6}\,\epsilon^2\,\mu_0\, a_{XXX} = O(\epsilon^3).} \tag{4.5}$$

What equation does a satisfy in terms of the variables $\boldsymbol{\chi = X - c_0 T}$ and $\boldsymbol{\tau = \epsilon^2\, T = \epsilon^3\, t}$?

Hint. Write $k = k_0 + \epsilon\,\kappa$ *and substitute this into (4.2–4.3). Then use the result in (4.1).*

5 Radiation damping #02

Statement: Radiation damping #02

Consider a semi-infinite shallow water channel, with a rectangular cross-section of width $\boldsymbol{w}$, and a paddle at the end. The paddle has mass $\boldsymbol{m}$, and it is kept in place by a spring — with spring constant $\boldsymbol{\kappa}$. In addition, a force $\boldsymbol{f = f(t)}$ is applied by the paddle. We model the system using the shallow water wave equations in the channel

$$h_t + (h\,u)_x = 0 \quad \text{for} \quad x > \sigma(t), \tag{5.1}$$
$$u_t + u\,u_x + g\,h_x = 0 \quad \text{for} \quad x > \sigma(t), \tag{5.2}$$

where $h = h(x,\, t)$ is the water depth, $u = u(x,\, t)$ is the flow velocity, $\boldsymbol{g}$ is the acceleration of gravity, and $x = \sigma(t)$ is the position of the paddle. At the paddle position, $x = \sigma$, we have

$$\dot{\sigma} \;=\; u, \tag{5.3}$$
$$m\,\ddot{\sigma} \;=\; -\kappa\,(\sigma - x_0) - \underbrace{\frac{1}{2}\,g\,\rho\,w\,h^2}_{p_w} + f, \tag{5.4}$$

where $\boldsymbol{\rho}$ is the density of water, p_w is the pressure[3] force by the water on the paddle, and x_0 corresponds to the equilibrium position for the spring — see (5.5). Note that a forcing done by moving the spring attachment point, so

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[2] The purpose of the pre-factor $1/\epsilon$ is so that the integral of U does not vanish as $\epsilon \to 0$.

[3] Hydrostatic equilibrium, with pressure variations in the air above the water neglected.

that the spring force has the form $-\kappa\,(\sigma - x_0 - \chi(t))$, is the particular case of (5.4) where $f = \kappa\,\chi$ — in fact, we can always write f in this form.

At equilibrium

$$\sigma = 0, \quad u = 0, \quad h = H = \text{constant}, \quad f = 0, \quad \text{and} \quad \kappa\,x_0 = \frac{1}{2}\,g\,\rho\,w\,H^2. \tag{5.5}$$

Assume now an "infinitesimal" force and "infinitesimal" perturbations from equilibrium,[4] with $h = H + \eta$, and **write linearized equations of motion for σ, η, and u.** The equations for η and u will apply for $x > 0$, with boundary conditions at $x = 0$ involving both f and σ. Introduce the velocity potential ϕ, with $u = \phi_x$ and $\eta = -\frac{1}{g}\,\phi_t$, and **write the equations in terms of ϕ and σ.**

For the situation where all the transient waves in the channel are gone, and the waves there are solely the product of the forcing f, **derive an ode for σ.** This ode will have a damping coefficient, due to radiated energy carried away by the waves. Finally, **write the solution for the case when $f = a\,e^{i\,\omega\,t}$ — where a and ω are constants.**

6 Reflected wave from an active boundary #01

Statement: Reflected wave from an active boundary #01

Calculate the reflected wave for the linear wave equation problem in figure 6.1. Note that:

1. *There are no waves on $x < 0$. The equation applies for $x > 0$ only.*

2. *There is a critical angle θ_c at which something very special happens.*
Find θ_c, and explain the physical meaning of what you found.

3. Show also that the model is stable: *none of the normal modes grows* — see remark **6.1**.

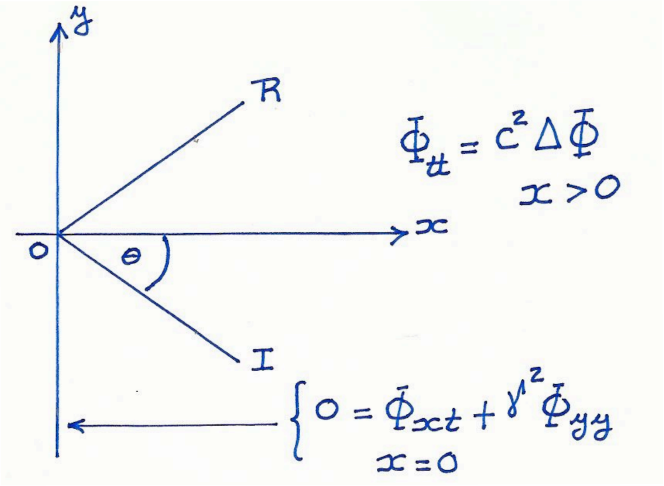

Figure 6.1: Reflected wave from an active boundary. Here c, $\gamma > 0$ are constants. The incident wave has the form $\boldsymbol{I = \exp\left(i\,k\,(-x\,\cos\theta + y\,\sin\theta - c\,t)\right)}$, with $0 < \theta < \pi/2$ and $k \neq 0$ constants.

The equation to be solved is

$$\Phi_{tt} - c^2\,\Delta\,\Phi = 0 \ \text{ for } \ x > 0, \quad \text{where } \ c > 0 \ \text{ is a constant,} \tag{6.1}$$

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[4] That is: u, σ, and η are "infinitesimal."

with the boundary condition

$$\Phi_{xt} + \gamma^2\, \Phi_{yy} = 0 \ \text{ at } \ x = 0, \quad \text{where } \ \gamma > 0 \ \text{ is a constant.} \tag{6.2}$$

This is a made-up *mathematical model for something that happens when there is an "active" boundary* — i.e.: extra energy is available there. The motivating example here is combustion, where the boundary is a plane detonation wave (where a chemical reaction occurs) and we look at the linearized problem near this solution. The "real" problem is far more complicated than the model here, with more waves (here only the acoustic waves have been allowed to survive), and more variables. But the basic wave phenomena that occurs is the same that you will find here.

Remark 6.1 *Because the equation and boundary condition are translational invariant in the y-direction, the problem can be Fourier Transformed in this variable, and we can write*

$$\Phi = \int_{-\infty}^{\infty} \phi(\ell,\, x,\, t)\, e^{i\,\ell\, y}\, d\ell, \quad \textit{where} \quad \phi = \frac{1}{2\,\pi} \int_{-\infty}^{\infty} \Phi\, e^{-i\,\ell\, y}\, dy \tag{6.3}$$

satisfies

$$\phi_{tt} - c^2\, \phi_{xx} + \ell^2\, c^2\, \phi = 0 \quad \textit{for} \quad x > 0, \tag{6.4}$$

and

$$\phi_{xt} - \gamma^2\, \ell^2\, \phi = 0 \quad \textit{at} \quad x = 0, \tag{6.5}$$

where $-\infty < \ell < \infty$.

The normal modes are solutions of this problem of the form $\phi = \varphi(\ell,\, x)\, e^{\lambda\, t}$, *decaying as* $x \to \infty$, *where* λ *is a constant. Since the equation for* φ *is a constant coefficients ode, it must be* $\varphi \propto e^{-\alpha\, x}$, *where* α *is a constant such that* $Re(\alpha) > 0$. ♣

7 Shallow water waves at the beach

Statement: Shallow water waves at the beach

Modulation theory for a slowly varying linear wave in a dispersive media states that the wave-number $\vec{k} = \nabla\, \theta$, and the wave-frequency $\omega = -\theta_t$, must satisfy the dispersion relation

$$G(\omega,\, \vec{k}) = 0, \tag{7.1}$$

where θ is the wave-phase. The wave-amplitude, in turn, satisfies the energy equation [5]

$$(a^2)_t + \operatorname{div}(\vec{c}_g\, a^2) = 0, \tag{7.2}$$

where $\vec{c}_g = \nabla_{\vec{k}}\, \omega$ is the group speed.

Example 1: For the Klein-Gordon equation $u_{tt} - c^2\, \Delta\, u + m^2\, u = 0$, where $c > 0$ and $m > 0$ are constants, $G = \omega^2 - c^2\, k^2 - m^2$ and $\vec{c}_g = (c^2/\omega)\, \vec{k}$. The theory works even if the waves are not dispersive, as in the case of linear shallow water

$$G = \omega^2 - g\, h\, k^2 \quad \text{and} \quad \vec{c}_g = (g\, h/\omega)\, \vec{k}, \tag{7.3}$$

where $h > 0$ is the water depth. ♣

If the media is slowly varying in space (changes happen on scales much larger than the wave-length) the theory still applies, with (7.1) replaced by

$$G(\omega,\, \vec{k},\, \vec{x}) = 0, \tag{7.4}$$

and $\vec{c}_g = \vec{c}_g(\vec{k},\, \vec{x})$ in (7.2).

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[5] The wave-energy flows at the group speed. Note that here we assume $a > 0$.

Example 2: Take $h = h(\vec{x})$ in (7.3). ♣

In particular, for single frequency waves, we can take $\theta = \phi(\vec{x}) - \omega\, t$, and $a = a(\vec{x})$, where ω is a constant. Then

$$G(\omega,\, \vec{k},\, \vec{x}) = 0 \quad \text{and} \quad \operatorname{div}(\vec{c}_g\, a^2) = 0, \tag{7.5}$$

where $\vec{k} = \nabla\, \phi$.

Problem tasks/questions: Consider the case of shallow water, in 1-D, for the situation where single frequency wave is approaching the shore at a gently sloped beach (thus $h = h(x)$ approaches zero slowly). Answer the following questions:

A. How do the wave-number and wave-length behave as h vanishes?

B. How does the wave-amplitude behave as h vanishes?

C. How does the maximum wave slope behave as h vanishes?

D. Does the linear approximation remain valid all the way to the shore?

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8 Slowly varying harmonic signaling for a string on a bed

8.1 Statement: Slowly varying harmonic signaling for a string on a bed

The equation for the (linear) vibrations of an homogeneous string under tension, over an homogeneous elastic bed, has the form

$$\rho\, u_{tt} - T\, u_{xx} + \kappa\, u = 0, \tag{8.1}$$

where $u = u(x,\, t)$ is the deviation from equilibrium of the string, and the (positive) constants ρ, T, and κ are the string density, the string tension, and the bed elastic constant, respectively.

In the lectures we analyzed the signaling problem, for a semi-infinite string $0 < x < \infty$, characterized by the boundary condition

$$u(0,\, t) = \operatorname{Re}\left(a\, e^{-i\,\Omega\, t}\right), \tag{8.2}$$

where a is a complex constant, and $\Omega > 0$ is a real constant. If $\boldsymbol{\Omega > \omega_c = \sqrt{\kappa/\rho}}$, this signaling problem has the general "steady state" solution

$$u(x,\, t) = \operatorname{Re}\left((a-b)\, e^{i\,(K\,x - \Omega\,t)} + b\, e^{i\,(-K\,x - \Omega\,t)}\right), \quad \text{with } K = \sqrt{(\rho/T)(\Omega^2 - \omega_c^2)} > 0, \tag{8.3}$$

where b is a constant. By calculating the average (over one time period $2\,\pi/\Omega$) energy flux produced by a plane harmonic wave — namely: $< -T\, u_t\, u_x >$ — we showed that **it must be $b = 0$,** since the component $\boldsymbol{b\, e^{i\,(-K\,x - \Omega\,t)}}$ in (8.3) corresponds to an energy flux from infinity towards $x = 0$.

This problem aims at arriving to the same result, but using a different approach. (8.4)

We begin by considering the a-dimensional version of the problem above, namely:

$$u_{tt} - u_{xx} + u = 0, \quad \text{for } x > 0, \tag{8.5}$$

with boundary condition

$$u(0,\, t) = \operatorname{Re}\left(a\, e^{-i\,\Omega\, t}\right), \quad \text{where } \Omega > 1 = \omega_c. \tag{8.6}$$

The corresponding wave number is then $\boldsymbol{K = \sqrt{\Omega^2 - 1}}$.

Next we generalize the problem to a situation where the forcing at the starting end of the string is **turned on slowly.** Namely, instead of taking a constant in (8.6), we assume that a is a slow[6] function of time

$$a = a(\tau), \quad \text{where } \tau = \epsilon\, t \quad \text{and } 0 < \epsilon \ll 1. \tag{8.7}$$

Because the amplitude $a(\tau)$ varies slowly, we expect that the solution to (8.5 – 8.7) will be, at least at leading order, close to the solution to the steady state that occurs when a is a constant — hence, it will have the same form. On the other hand, because the forcing amplitude is not constant, we expect that the amplitude of the solution to (8.5 – 8.7) will change slowly. Furthermore, the speed of propagation of changes in the forcing is bounded,[7] hence the amplitude of the solution to (8.5 – 8.7) cannot be independent of x, as this would require infinite speed of propagation. But, because the forcing varies slowly, a finite speed of propagation suggests that the space dependence should also be slow, as the amplitude has time to adjust (anywhere) to some sort of local steady state amplitude.

The arguments in the prior paragraph are somewhat vague, but they suggest that we should look for solutions of (8.5 – 8.7) of the form

$$\boldsymbol{u \approx \mathrm{Re}\left(A(\chi,\, \tau)\, e^{i\,(K\,x - \Omega\,t)}\right), \quad \text{where } \chi = \epsilon\, x.} \tag{8.8}$$

TASK #1: Seek such solutions, and **find the equation that $A = A(\chi,\, \tau)$ must satisfy.**

Hint. *The idea is that the right hand side in (8.8) should be a solution up to some small error. Hence write*

$$u = \mathrm{Re}\left(A(\chi,\, \tau)\, e^{i\,(K\,x - \Omega\,t)}\right) + \epsilon\, u_1(x,\, t;\, \chi,\, \tau) + \dots \tag{8.9}$$

substitute into the equation, and find an equation that u_1 must satisfy as a function of x and t. This equation will be forced by terms produced by the χ and τ dependence of A. Now select this dependence in such a way that no growing component in u_1 is triggered by the forcing — if u_1 grows, then the error in the approximate solution in (8.8) will not be small, so that (8.8) will not really be an approximate solution.

TASK #2: Perform the same analysis, but now look for solutions of (8.5 – 8.7) of the form

$$u \approx \mathrm{Re}\left(B(\chi,\, \tau)\, e^{i\,(-K\,x - \Omega\,t)}\right), \tag{8.10}$$

and find the equation that B satisfies.

TASK #3: The equations that you find for A and B will be fairly simple, and you will be able to write their general solution explicitly. By looking at these solutions, *argue that (8.9) is an acceptable solution to the problem in (8.5 – 8.7), but (8.10) is not.* This is the result promised in (8.4).

TASK #4: What role does the group speed play in all of this? Where does it appear?

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9 String on an elastic bed: harmonic forcing at the critical frequency

Before doing this problem, check the *Lecture topics for 18376* notes: Section: Radiation damping. Subsection: Semi-infinite string over elastic bed with mass-spring at end. Subsubsection: Harmonic forcing.

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[6] The reason we **need to use a-dimensional variables** in this problem is that, otherwise, “small” has no meaning.

[7] (8.5) is hyperbolic, with characteristic speeds ±1. It can be shown that no signal propagates faster than 1.

Statement: String on an elastic bed: harmonic forcing at the critical frequency

Consider a semi-infinite string over an elastic bed, under tension, with a forced mass-spring system attached at its end (assume also small, in-plane, motion). With properly selected a-dimensional variables, the equations are

$$\begin{aligned} u_{tt} - u_{xx} + u &= 0 \qquad \text{for} \quad x > 0, && (9.1) \\ u_{tt} + \Omega^2\, u &= 2\,\nu\, u_x + G \qquad \text{at} \quad x = 0, && (9.2) \end{aligned}$$

where Ω and ν are positive constants, $G = G(t)$ is the force applied to the mass attached to the string, and $2\nu u_x(0,\, t)$ is the force by the string (due to its tension) on the mass. We will make the following assumptions:

a1. The forcing is harmonic, specifically: $\boldsymbol{G = e^{i\,\omega\,t}}$, with $\boldsymbol{0 < \omega < 1}$. (9.3)

a2. The following applies: $\boldsymbol{\Omega^2 < 1}$. (9.4)

A particular solution to (9.1–9.3) is given by $\boldsymbol{u_p = a\, e^{i\,\omega\,t - \ell\,x}}$, with $\boldsymbol{a = 1/(\Omega^2 + 2\nu\ell - \omega^2)}$, (9.5)
where $\boldsymbol{\ell = \sqrt{1-\omega^2}}$.
Below we show how to use this solution to generate the solution to the initial value problem for (9.1–9.3). However, note: **there is a critical value of $\boldsymbol{\omega}$, $\boldsymbol{\omega_c}$, at which (9.5) fails** — the value such that $\Omega^2 + 2\nu\ell - \omega^2 = 0$.

Remark 9.1 $\boldsymbol{\Omega^2 + 2\nu\ell - \omega^2 = 0}$ has exactly one solution for $\boldsymbol{0 < \omega < 1}$, $\boldsymbol{\omega_c}$.
Proof. Let $f(\omega) = \omega^2 - 2\nu\ell$. This function is increasing, and satisfies $f(0) = -2\nu < \Omega^2$ and $f(1) = 1 > \Omega^2$. ♣

Problem task: Find a particular solution for the case $\boldsymbol{\omega = \omega_c}$.

Hint. *Use the technique illustrated by the following example:* Consider the ode: $\boldsymbol{\ddot{y} + y = e^{i\,\omega\,t}}$ [A]. This has the solution $y_p = (1-\omega^2)^{-1} e^{i\,\omega\,t}$, valid as long as $\omega^2 \neq 1$. To find a solution for $\omega = 1$, notice that $z = e^{i\,\omega\,t}$ satisfies, for any ω, the ode: $\ddot{z} + z = (1-\omega^2)\, e^{i\,\omega\,t}$ [B]. Now let $\xi = \frac{\partial z}{\partial \omega}$, and take the derivative of [B] with respect to ω. This yields the equation: $\ddot{\xi} + \xi = -2\,\omega\, e^{i\,\omega\,t} + i\,t\,(1-\omega^2)\, e^{i\,\omega\,t}$ [C]. Evaluating now [C] at $\omega = 1$ leads to the desired particular solution, specifically: $\boldsymbol{y_p = -\frac{1}{2}\,\xi(\omega = 1) = -\frac{1}{2}\, i\, t\, e^{it}}$.
Note also that the answer to this problem is just slightly longer than this hint. ♣

From particular to the general solution.

Here we show how to use a particular solution of (9.1–9.3), to reduce the initial value problem to one that can be solved by ode techniques and Fourier Transforms. Note: you do not need to read this to do the problem!
Disclaimer: the approach presented below is probably not *"the best"*. Think of it as a *proof of concept* only.

We begin by writing $\boldsymbol{u = u_p + w}$, where **$\boldsymbol{w}$ solves (9.1–9.2) with $\boldsymbol{G = 0}$ and initial data:**

$$\boldsymbol{w(x,\, 0) = w_0(x) = u(x,\, 0) - u_p(x,\, 0)} \text{ and } \boldsymbol{w_t(x,\, 0) = w_1(x) = u_t(x,\, 0) - (u_p)_t(x,\, 0)}. \qquad (9.6)$$

Introduce now $v = v(x,\, t)$ by
where the operator $\mathcal{L}$ is defined
by the equation. Then

$$\boldsymbol{v = \mathcal{L}\, w = w_{xx} - (1-\Omega^2)\, w - 2\,\nu\, w_x}, \qquad (9.7)$$

$$\boldsymbol{v_{tt} - v_{xx} + v = 0} \quad \text{for} \quad x > 0, \quad \text{with } \boldsymbol{v(0,\, t) = 0}. \qquad (9.8)$$

The initial conditions for this equation $\Big(\boldsymbol{v(x,\, 0) = v_0(x)}$ and $\boldsymbol{v_t(x,\, 0) = v_1(x)}\Big)$ follow from (9.6–9.7).
Why is it $\boldsymbol{v(0,\, t) = 0}$? This is because w satisfies (9.1), so that $\boldsymbol{v = w_{tt} + \Omega^2\, w - 2\,\nu\, w_x}$ as well.
Then

$$v = \int_0^\infty \left(\hat{v}_0(k)\, \sin(k\,x)\, \cos\left(\sqrt{1+k^2}\, t\right) + \hat{v}_1(k)\, \sin(k\,x)\, \frac{\sin\left(\sqrt{1+k^2}\, t\right)}{\sqrt{1+k^2}} \right) \mathrm{d}k, \qquad (9.9)$$

where $\hat{v}_0$ and $\hat{v}_1$ are the sine-Fourier Transforms of v_0 and v_1.

The issue is now: **Given $\boldsymbol{v}$, how do we recover $\boldsymbol{w}$?** To
do this we observe that, from the definition of v, we have $\boldsymbol{\mathcal{L}\, w = v}$. (9.10)
Thus

$$w(x,\, t) = w_1(x,\, t) + \alpha(t)\, e^{\lambda_1\, x}, \quad \text{where} \quad w_1(x,\, t) = \int_0^\infty G(x,\, y)\, v(y,\, t)\, \mathrm{d}y, \qquad (9.11)$$

α is a function to be determined, λ_1 is defined below, and G is the Green's function for $\mathcal{L}^{-1}$ with zero boundary condition at $x = 0$. That is:

$$G = \frac{1}{\lambda_1 - \lambda_2}\left(e^{h\,(x-y)} - e^{\lambda_1 x - \lambda_2\, y}\right) \tag{9.12}$$

where $h = \lambda_2$ if $x < y$, $h = \lambda_1$ if $x > y$,

$$\lambda_1 = -\nu - \sqrt{\nu^2 + (1-\Omega^2)} < 0,$$

and

$$\lambda_2 = -\nu + \sqrt{\nu^2 + (1-\Omega^2)} > 0.$$

The λ_j are the two roots of $\lambda^2 - 2\,\nu\,\lambda = 1 - \Omega^2$, the characteristic equation for $\mathcal{L}$.

Why (9.11)? Because $\mathcal{L}\,w = v$ determines w up to an homogeneous solution, but $e^{\lambda_2\,x}$ is not allowed because $\lambda_2 > 0$.

Now, because v satisfies (9.1), and $\mathcal{L}\,w_1 = v$, we have $\mathcal{L}((w_1)_{tt} - (w_1)_{xx} + w_1) = 0$, hence

$$(w_1)_{tt} - (w_1)_{xx} + w_1 = \beta(t)\,e^{\lambda_1\,x}. \tag{9.13}$$

But both v and w_1 vanish at $x = 0$. Hence evaluating (9.10) and (9.13) at $x = 0$ we obtain: $(w_1)_{xx} = 2\,\nu\,(w_1)_x$ and $(w_1)_{xx} = -\beta$. Thus

$$\boldsymbol{\beta(t) = -2\,\nu\,(w_1)_x(0,\,t)}. \tag{9.14}$$

Finally, substituting $w = w_1 + \alpha(t)\,e^{\lambda_1\,x}$ into $w_{tt} - w_{xx} + w = 0$, and using (9.13), yields an **equation that determines $\boldsymbol{\alpha}$.** That is

$$\boldsymbol{\ddot{\alpha} + (1 - \lambda_1^2)\,\alpha + \beta = 0}. \tag{9.15}$$

The task of finding how to get initial conditions for this equation is left to the reader. Note that w, as defined by all these steps, satisfies the boundary condition at $x = 0$. Why? Because using $w_{tt} - w_{xx} + w = 0$ in $\mathcal{L}\,w = v$ yields $v = w_{tt} + \Omega^2\,w - 2\,\nu\,w_x$, and v vanishes at $x = 0$.

Remark 9.2 Provided that the initial data are reasonably smooth: as $t \to \infty$, v vanishes; consequently, w as well. It follows that: **As $\boldsymbol{t \to \infty}$, the solution to (9.1–9.3) is dominated by the particular solution, $\boldsymbol{u \sim u_p}$.** ♣

Here you may wonder: *wait a second, the particular solution is not unique; what if I use a different one from the one above?* The answer is that it does not matter: the difference between any two particular solutions vanishes.

THE END.