18.376 Problem Set #01, MIT Spring 2023

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March 22, 2023

Due: Thu April 6, 2023.

Turn it in pdf format via the Canvas web page.

Note. Answers to all the problems will be posted, but *only some problems will be graded.*
The graded problems are "quiz #01".

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1 The flux for a conserved quantity must be a vector

Statement: The flux for a conserved quantity must be a vector

Note: Below (for simplicity) we present many arguments/questions in 2D. However they apply just as well in nD; $n = 3,\,4,\,\ldots$

Consider some conserved quantity, with density $\rho = \rho(\vec{x}, t)$ and flux vector $\vec{q} = \vec{q}(\vec{x}, t)$ in 2D. Then, in the absence of sources or sinks, we made the argument that conservation leads to the (integral) equation

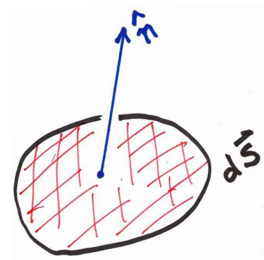
$$\frac{d}{dt}\int_{\Omega} \rho\, \mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\partial\Omega} \vec{q}\cdot\hat{n}\,\mathrm{d}s, \tag{1.1}$$

for any region Ω in the domain where the conserved "stuff" resides, where $\partial\Omega$ is the boundary of Ω, s is the arc-length along $\partial\Omega$, and $\hat{n}$ is the outside unit normal to $\partial\Omega$.

There are **two implicit assumptions** used above

1. The flux of conserved stuff is local: stuff does not vanish somewhere and re-appears elsewhere (this would not violate conservation). For most types of physical stuff this is reasonable. But one can think of situations where this is not true — e.g.: when you wire money, it disappears from your local bank account, and reappears elsewhere (with some loses due to fees, which go to other accounts).
2. The flux is given by a vector. But: $\Rightarrow$ *Why should this be so?* $\Leftarrow$ (1.2)
   The objective of this problem is to answer this question.

Given item **1**, **the flux can be characterized/defined as follows** †



For any surface element $d\vec{S}$ (at a point $\vec{x}$, with unit normal $\hat{n}$) the flux indicates how much stuff, per unit time and unit area, crosses $d\vec{S}$ from one side to the other, in the direction of $\hat{n}$. } (1.3)

† This is in 3D. For a 2D, change "surface element" to "line element".

It follows that the flux should be a scalar function of position, time, and direction. That is:

$$q = q(\vec{x},\, t,\, \hat{n}), \tag{1.4}$$

where q is the amount of stuff, per unit time and unit length, crossing a curve [1] with unit normal $\hat{n}$ from one side to the other (with direction [2] given by $\hat{n}$). Then (1.1) takes the form

$$\frac{d}{dt}\int_{\Omega} \rho \, \mathrm{d}x_1 \mathrm{d}x_2 = -\int_{\partial\Omega} q(\vec{x},\, t,\, \hat{n})\, \mathrm{d}s. \tag{1.5}$$

However, in this form we cannot use Gauss' theorem to transform the integral on the right over $\partial\Omega$, into one over Ω. *This is a serious problem, for this is the crucial step in reducing* (1.1) *to a pde.*

Your task: *Show that,* provided that ρ and q are "nice enough" functions (e.g.: continuous partial derivatives), *equation* (1.5) *can be used to show that* q *has the form*

$$q = \hat{n} \cdot \vec{q}, \tag{1.6}$$

for some vector valued function $\vec{q}(\vec{x},\, t)$.

Hints.

A. *It should be obvious that the flux going across any curve (surface in 3D) from one side to the other should be equal and of opposite sign to the flux in the opposite direction. That is, q in (1.4) satisfies*

$$q(\vec{x},\, t,\, -\hat{n}) = -q(\vec{x},\, t,\, \hat{n}). \tag{1.7}$$

Violation of this would result in the conserved "stuff" accumulating (or being depleted) at a finite rate from a region with zero area (zero volume in 3D), which is not compatible with the assumption that ρ_t is continuous and equation (1.5). **Note:** there are situations where it is reasonable to make models where conserved "stuff" can have a finite density on curves or surfaces (e.g.: surfactants at the interface between two liquids, surface electric charge, etc.). Dealing with situations like this requires a slightly generalized version of the ideas behind (1.7).

B. *Given an arbitrary small curve segment* [3] *of length $h > 0$ and unit normal $\hat{n}$, realize it as the hypotenuse of a right triangle where the other sides are parallel to the coordinate axes. Then write (1.5) for the triangle, divide the result by h, and take the limit $h \downarrow 0$. Note that, if the segment of length h is parallel to one of the coordinate axis, then one of the sides of the triangle has zero length, and the triangle has zero area — but the argument still works, albeit trivially (it reduces to the argument in* **A**).

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[1] In 3D: "... and unit area, crossing a surface ..."
[2] A positive q means that the net flow is in the direction of $\hat{n}$.
[3] You can assume it is straight, since a limit $h \downarrow 0$ will occur.

2 Small transversal vibrations of a beam

Statement: Small transversal vibrations of a beam

A beam is a structure where one dimension (the axial dimension) is much larger than the other two (the transversal dimensions), see item **c**. In this problem you are asked to derive an equation for the small transversal vibrations of an homogeneous elastic beam with (constant) rectangular cross section, which is not under tension or compression. Further simplifying assumptions are:

a. The wavelength of the vibrations is much bigger than the transversal dimensions of the beam.

b. The vibrations are in-plane. This means that the motion of the beam is restricted to the plane determined by its axis, and the direction of one of the sides of the rectangular cross section.
Think of a blade. When the two transversal dimensions are very different, it is much harder to excite vibrations along the larger direction.

Under the conditions stated above, the beam motion can be described in terms of the *position of the beam axis* $y = u(x,\, t)$. Let $\boldsymbol{\rho} =$ constant $> \mathbf{0}$ be the *mass per unit length of the beam.* Then:

Task #1 of 5. Use conservation of the transversal momentum to derive an equation for $\boldsymbol{u}$.

Task #2 of 5. Use conservation of energy to derive another equation for $\boldsymbol{u}$.

Task #3 of 5. Show that the solutions to the task #1 equation satisfy the task #2 equation.

To do the problem you will need a few things from § 2, as follows:

— To do task #1 you need (fo.8), where f_s is defined in item **5** — E and I are constants.

— In addition, to do task #2, you need (fo.6) and (fo.9), where τ_b is defined in item **6**.

— The summary of facts in (fo.10) may be useful.

You do not need anything else, but I strongly encourage you to read, and understand, § 2.

Hint for task #2. Do not forget that energy flow is not only produced by forces [force times velocity], but by torque as well [torque times angular velocity]

More tasks

Task #4 of 5. Show that the task #1 equation yields conservation of angular momentum.

Task #5 of 5. Take a thin steel blade with a rectangular cross section (e.g., the blade from a metal hand saw, see the picture). Clamp one end using a bench vise, and leave the other end free. For the equation in task #1, in this situation, what boundary conditions on u should be imposed at each end?

What if there is a frictionless-hinge at each end of the blade, that keeps the end fixed, but allows the blade to freely rotate there. What boundary conditions should be used in this case?

Hints for task #4. (1) Recall that the angular momentum of a mass point moving in a plane[4] is given by $A = \pm\, m\, v\, d$, where: m is the mass; v is the speed; d is the distance from the straight line through the point along the direction of motion, to some fixed point in space; and the sign is positive if the mass is moving counter-clockwise around the selected point in space. For example, if the point in space is the origin, and the point path is $x = a =$ constant and $y = v\, t$, then $A = m\, a\, v$. To do task #4, take the selected point to be the origin of coordinates. (2) Recall also that angular momentum is produced by torque, and that torque can manifest in two ways: "directly" (as in the torque applied through the axle to a wheel), or through forces (as when you rotate a wheel by pushing through the edge): a wheel-chair can either have a motor, or the user can move the wheels with his/her hands, or both (or someone else can push the chair).

Side remarks

[4] In general the angular momentum is a vector, but for in-plane motion it is a scalar.

c. In a string the transversal dimensions are neglected (thus a string has no bending resistance). On the other hand, for a beam they are assumed small, but their effect is not neglected (thus a beam can support a transversal load).

d. An elastic beam can support vibrations along each of the two transversal directions, as well as longitudinal vibrations. It can support torsional vibrations (twist along the axis) as well. In principle it can also support torsion along the transversal directions — but these are not consistent with the beam approximation (Euler-Bernoulli assumptions, see § 2).

Euler-Bernoulli beam theory states that the beam cross sections move as rigid planes when the beam vibrates, and remain normal to the beam axis (or, in 2-D, the beam center plane, in red here). Thus the beam motion can be described in terms of the axis behavior.

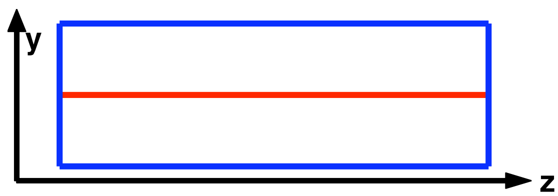

Figure 2.1: Cross section of a (rectangular) vibrating beam, of width w and height h.

The blue lines are the edge of the beam. The red line is the beam axis. The green lines are typical "fibers" — see paragraph above (fo.5). The magenta lines are typical beam cross sections, which move as rigid planes, and remain normal to the axis and fibers.

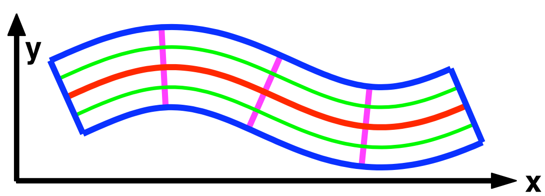

Figure 2.2: Side view of a beam undergoing in-plane motion, as per Euler-Bernoulli theory.

Beam elastic energy, shear force, and toque

This subsection's purpose is to provide contextual information needed to understand and do the problem. It does not include any further tasks to be done.

For beams under deformations that are not too large (basically, the situation in item **a** above), the Euler-Bernoulli assumptions apply [5] (see figures 2.1 and 2.2)

eb1. Cross sections of the beam do not deform in a significant manner under the application of transverse or axial loads, and can be assumed as rigid.

eb2. During deformation, the cross section of the beam remains planar, and normal to the deformed axis of the beam.

From this assumptions, and item **b**, it follows that we can describe the motion of the beam using just two 1D functions (see (fo.1) below, as follows:

1. Take a coordinate system such that the beam at equilibrium is $0 < x < L$, $|y| < \frac{1}{2}\,h$, and $|z| < \frac{1}{2}\,w$, where L is the beam length, h is its height, and w is its width. Here x is the axial coordinate, and the motion is in the x-y plane (no dependence on z).

2. We label each mass-element in the beam by its $(x,\, y,\, z)$ coordinates at equilibrium, and describe the beam at any time by giving the coordinates of each mass-element as a function of time and the equilibrium coordinates, that is: $X = X(x,\, y,\, t)$, $Y = Y(x,\, y,\, t)$, $Z = z$, where we have used item **b** to simplify the dependence on z.

[5] These assumptions have been extensively confirmed for (solid cross section) slender beams made of isotropic materials.

3. Let $\boldsymbol{v = v(x,\, t) = X(x,\, 0,\, t) - x}$ and $\boldsymbol{u = u(x,\, t) = Y(x,\, 0,\, t)}$ be the two *functions describing the motion of the beam axis.* Then, from eb1 and eb2,

$$\left.\begin{array}{rcl} X & = & x + v - \frac{1}{d}\, y\, u_x, \\ Y & = & u + \frac{1}{d}\, y\,(1 + v_x), \end{array}\right\} \text{ where } d = \sqrt{(1+v_x)^2 + u_x^2}. \tag{fo.1}$$

This follows because the unit normal vector to each planar cross section of the beam (which moves as a rigid body, as per eb1–eb2) is given by

$$\hat{n} = \frac{1}{d}\,(1 + v_x,\, u_x,\, 0)^T \quad \Longrightarrow \quad \hat{t} = \frac{1}{d}\,(-u_x,\, 1 + v_x,\, 0)^T. \tag{fo.2}$$

Here $\hat{t}$ is the unit vector tangent to the planar cross section, in the $z = 0$ plane, pointing towards $y > 0$ (since $1 + v_x > 0$, because v_x is small, as explained below).

Detail: $\hat{n}$ is the unit tangent vector to the beam axis: $X = x + v$, $Y = v$, and $Z = 0$.

4. Finally, the condition of *small vibrations for a beam which is not under tension/compression* translates into

$$\text{Both } u_x \text{ and } v_x \text{ are small. Furthermore: } v_x = O(u_x^2). \tag{fo.3}$$

Thus *we neglect quadratic terms in u_x, and write*

$$X = x - y\, u_x \quad \text{and} \quad Y = y + u. \tag{fo.4}$$

The assumption here is that the deviations of the beam shape from horizontal and straight are small. Under this condition, in the absence of (significant) axial stretching or compression, the changes in horizontal dimensions of the beam cannot be larger than quadratic in u_x..

Because the beam cross sections do not deform, and behave as rigid surfaces, [6] all the deformation occurs along the curves $y =$ constant and $z =$ constant, [7] which are either stretched or compressed. We can thus obtain the *elastic energy* in the beam by computing the elastic energy in each fiber, and integrating over all of them. Along each fiber the change in arclength (relative to equilibrium) is

$$\Delta L\, ds = \left(\sqrt{X_x^2 + Y_x^2} - 1\right)\, dx = -y\, u_{xx}\, dx, \tag{fo.5}$$

upon use of (fo.3–fo.4). From remark **2.1**, $\frac{1}{2}\, E\, dz\, dy\, y^2 \int u_{xx}^2\, dx$ is the elastic energy in each fiber. Thus

$$\mathcal{V} = \frac{1}{2}\, E\, I \int u_{xx}^2\, dx = \textbf{beam elastic energy}, \quad \text{where } I = \frac{1}{12}\, h^3\, w \tag{fo.6}$$

is the *second moment of the beam's cross section* $I = \iint y^2\, dz\, dy$.

Next we will use (fo.6) to compute both the **shear force,** $\boldsymbol{f_s}$, and the **torque,** $\boldsymbol{\tau_b}$, along the beam:

5. At any point along the beam, $\boldsymbol{f_s = f_s(x,\, t)}$ **is the force** in the y-direction (transversal) that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite force is applied by the right section on the left section). If the beam were to be cut, these are the forces that would be needed to keep in position the lips of the cut.

[6] This is an approximation. There is some deformation and there are forces. But they are small and we neglect them.
[7] We will call these curves "fibers".

6. At any point along the beam, $\boldsymbol{\tau_b = \tau_b(x,\, t)}$ **is the torque** that the section of the beam to the left of the point applies on the section to the right of the beam (the opposite torque is applied by the right section on the left section). If the beam were to be cut, these are the torques that would be needed to keep the beam ends at the cut from rotating.

Let us now *investigate how $\mathcal{V}$ changes as we deform the beam, from some configuration to another.* To do this we need to apply forces to the beam, which do work against the opposite forces generated by the elastic deformation of the beam (this is how the energy changes). Thus, by looking at how the energy changes as the beam is deformed, we can ascertain the elastic forces in the beam for any given configuration.

Hence assume that $u = u(x,\, \tau)$, where τ is a parameter that we use to describe the successive configurations of the beam as its shape changes.[8] Then, for any arbitrary interval $a < x < b$, the energy within the interval varies as follows

$$\begin{aligned} \frac{d\mathcal{V}}{d\tau} &= E\,I \int_a^b u_{xx}\, u_{xx\tau}\, dx \qquad\qquad \text{(integrate by parts twice)} \\ &= E\,I\,(u_{xx}\, u_{x\tau})\Big|_a^b - E\,I\,(u_{xxx}\, u_\tau)\Big|_a^b + E\,I \int_a^b u_{xxxx}\, u_\tau\, dx. \end{aligned} \tag{fo.7}$$

This equation tells us that

7. The terms $E\,I\,u_{xxx}\,u_\tau$ at $x = a$, and $-E\,I\,u_{xxx}\,u_\tau$ at $x = b$, are the work (per unit τ-time) done by the applied forces to move the ends of the beam at the velocity u_τ of each end. This means that forces $E\,I\,u_{xxx}$ and $-E\,I\,u_{xxx}$ must be applied there. These must be the forces applied by the beam regions on each side of the interval. We conclude that $$\boldsymbol{f_s = E\,I\,u_{xxx}}. \tag{fo.8}$$

8. Since $u_{x\tau}$ is the rate of change in beam angle, and work is also torque times angle, an argument entirely similar to the one in **7** shows that $$\boldsymbol{\tau_b = -E\,I\,u_{xx}}. \tag{fo.9}$$

9. The last term in (fo.7) yields the force per unit length that must be applied along the beam to cause the deformation: $-E\,I\,u_{xxxx}$. The force per unit length needed to keep the beam with the given shape, and equilibrate the force $E\,I\,u_{xxxx}$ with which the beam pushes back.

Note that the forces producing beam deformation need not be "external forces" applied to the beam. When the beam is vibrating, the forces involved are those caused by the inertia of the beam.

Summary of useful facts. (fo.10)

$\boldsymbol{u}$ is the beam transversal deformation. The beam axis is $y = u(x, t)$ (small vibrations).
$\boldsymbol{u_t}$ is the beam transversal velocity.
$\boldsymbol{u_x}$ is the beam angle.
$\boldsymbol{u_{xt}}$ is the beam angular velocity.
$\boldsymbol{\tau_b}$ is the torque (see item **6**), given by $\tau_b = -E\,I\,u_{xx}$.
$\boldsymbol{f_s}$ is the shear force (see item **5**), given by $f_s = E\,I\,u_{xxx}$.
The elastic energy density is $\frac{1}{2}\,E\,I\,u_{xx}^2$.
I is the second moment of the beam's cross section. given by $I = \iint y^2\, dz\, dy$.

Remark 2.1 What is the energy stored in a slightly stretched/compressed elastic thin string?

> *The calculation below is valid as long as Hooke's law applies. In addition, we neglect any changes in the area of the string cross section when under tension or compression. In fact, the Euler-Bernoulli assumptions require that changes of this type be ignored.*

[8] The idea is that we deform the beam very, very, slowly. Thus, at every moment the applied forces, and the elastic forces generated by the deformation equilibrate each other exactly. There is no kinetic energy; all the work done by the applied forces go into the beam's elastic energy. Thus τ is a very "special" time.

First consider the energy stored in a short straight segment of length $L(t)$, as we stretch/compress it from length $L_0 = L(0)$ to $L_1 = L(T)$, where L_0 is the equilibrium length. From Hooke's law, the elastic force is

$$F = E\,\frac{L(t)-L_0}{L_0}\,A, \quad \textit{where } A = \textit{area of string cross section}, \tag{fo.11}$$

and $\boldsymbol{E} =$ **Young's modulus** *(E has units of force over area). The energy at the end of the process is*

$$\mathcal{V}_{\text{seg}} = \int_0^T F(t)\,\frac{dL}{dt}\,dt = \frac{1}{2}\,E\,A\,\frac{(L_1-L_0)^2}{L_0}. \tag{fo.12}$$

Note that this formula depends only on L_1 and L_0, not how we go from one to the other. Think now of the whole string as composed of a sequence of infinitesimal straight segments. Then we can use (fo.12) to write

$$\mathcal{V}_{\text{str}} = \frac{1}{2}\,E\,A\int\left(\sqrt{\dot{X}^2+\dot{Y}^2}-1\right)^2 ds, \tag{fo.13}$$

where points on the string is labeled by s (the arclength along the string at equilibrium), the string is given by $x = X(s)$ and $y = Y(s)$, and the dots indicate derivatives with respect to s.

Detail: Take, in (fo.12), $L_0 = ds$. Then $L_1 = \sqrt{\dot{X}^2+\dot{Y}^2}\,ds$, and $\mathcal{V}_{\text{seg}} = \frac{1}{2}\,E\,A\left(\sqrt{\dot{X}^2+\dot{Y}^2}-1\right)^2 ds$.

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3 Kelvin waves in a rotating basin

Statement: Kelvin waves in a rotating basin

Waves trapped near a boundary can occur via a wave-guide effect, when the waves velocity is lower near the boundary than elsewhere; e.g.: edge waves. Rotation can do the same. The simplest example exhibiting this behavior are the (linear) shallow water waves with constant rotation (f-plane approximation)

$$\boldsymbol{\eta_t + h\,(u_x + v_y) = 0,} \tag{3.1}$$

$$\boldsymbol{u_t + g\,\eta_x = f\,v,} \tag{3.2}$$

$$\boldsymbol{v_t + g\,\eta_y = -f\,u,} \tag{3.3}$$

where η is the surface deviation from equilibrium, $h > 0$ is the (constant) depth, u and v are the two components of the flow velocity, g is the acceleration of gravity, and $f \neq 0$ is the (constant) rotation angular velocity.
Consider these equations in the region $\boldsymbol{y > 0}$, with **boundary condition $v = 0$ at $y = 0$,** and **search for (non-trivial) solutions that**
(i) Are functions of $(x - s\,t)$, and y only, where s is some constant.
(ii) Are exponentially trapped near the $y = 0$ boundary.

Hint. *Look for solutions of the form $\eta = N(x-s\,t)\,e^{-\lambda\,y}$, $u = U(x-s\,t)\,e^{-\lambda\,y}$, etc., where s and λ are constants such that* $\text{Re}(\lambda) > 0$. Eliminate from consideration solutions where N and U are constants — that is: "trivial" solutions.

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4 Amplitude modulation with frozen carrier frequency

Statement: Amplitude modulation with frozen carrier frequency

Consider a dispersive equation in n-D of the form

$$\boldsymbol{u_t + i\,\Omega(-i\,\nabla)\,u = 0,} \tag{4.1}$$

where: (i) $u = u(\vec{x},\,t)$ is a scalar function, (ii) ∇ is the gradient operator in n-D, and (iii) $\omega = \Omega(\vec{k})$ is a smooth, real valued (scalar), of the wave-number vector $\vec{k} \in \mathcal{R}^n$. †

† If you want to, you may assume that Ω is a polynomial: a finite sum $\boldsymbol{\Omega = \sum a_{i_1 i_2 \dots i_n} k_1^{i_1} k_2^{i_2} \dots k_n^{i_n}}$, for some coefficients $\boldsymbol{a_{i_1 i_2 \dots i_n}}$. However, this is not actually necessary.

This equation is dispersive, with dispersion relation $\omega = \Omega(\vec{k})$.

Consider now solutions of the form

$$\boldsymbol{u \sim A(\vec{\chi},\, \tau)\, e^{i\,\theta}}, \tag{4.2}$$

with $\boldsymbol{\vec{\chi} = \epsilon\,\vec{x}}$, $\boldsymbol{\tau = \epsilon\, t}$, and $\boldsymbol{\theta = \vec{k}_0 \cdot \vec{x} - \omega_0\, t}$, where: (i) $0 < \epsilon \ll 1$ is a constant, (ii) $\vec{k}_0$ is a constant wave-number vector, and (iii) $\omega_0 = \Omega(\vec{k}_0)$.

Derive the leading order equation satisfied by A.

Hint. *Think of u as a function of $\vec{x}$ and $\vec{\chi}$, with $\vec{x}$ and $\vec{\chi}$ independent variables (same for t and τ). Then the equation can be written in the form*

$$\boldsymbol{u_t + \epsilon\, u_\tau + i\,\Omega(-i\,\nabla_1 - i\,\epsilon\,\nabla_2)\, u = 0}, \tag{4.3}$$

where ∇_1 is the gradient operator for $\vec{x}$ and ∇_2 is the gradient operator for $\vec{\chi}$. Now expand the operator $\Omega(-i\,\nabla_1 - i\,\epsilon\,\nabla_2)$ in powers of ϵ, and notice that evaluating functions of $-i\,\nabla_1$ is easy — just replace $-i\,\nabla_1$ by $\vec{k}_0$ (do you see why this?).

Remark 4.1 Consider a function, $f = f(x)$, evaluated on an operator (or matrix) $f = f(A)$. † When doing this we need to keep in mind that, because operators do not commute, the "regular" rules of calculus may not apply. For example: (i) Generally we can write $\frac{d}{dt} f(A(t)) = f'(A)\frac{dA}{dt}$ only if A and $\frac{dA}{dt}$ commute. (ii) Generally we can "Taylor" expand $f(A+B)$, for B small, only if A and B commute. ♣

† This can certainly be done when $\boldsymbol{f}$ is a polynomial. But this is not the only case: For example, if $\boldsymbol{f}$ is analytic, we can use Cauchy's theorem to write $\boldsymbol{f(A) = \frac{1}{2\pi i}\oint (z - A)^{-1} f(z)\,dz}$, where the contour of integration includes the spectrum of A.

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5 Two strings connected by a third one

Statement: Two strings connected by a third one

Consider two semi-infinite strings under tension, connected by a short piece of another string. If $u = u(x,\, t)$ denotes the transverse deviations of the strings, in the linear (small deviations and small slopes), the equation describing the system is

$$0 = u_{tt} - c^2\, u_{xx} \tag{5.1}$$

where $\boldsymbol{c = c_1}$ for $\boldsymbol{x < 0}$, $\boldsymbol{c = c_2}$ for $\boldsymbol{0 < x < L}$, $\boldsymbol{c = c_3}$ for $\boldsymbol{L < x}$, the $\boldsymbol{c_j > 0}$ are constants,[9] and $\boldsymbol{L}$ is the length of the connecting piece. This equation must be supplemented by the conditions

$$u \quad \text{and} \quad u_x \quad \text{are continuous across the joints at } x = 0 \text{ and } x = L. \tag{5.2}$$

Assume now that a monochromatic wave of wave-frequency ω arrives at the (composite) junction. Then:

1. **Calculate the reflection and transmission coefficients across the (composite) junction.** That is, find solutions to the equation of the following form:

$$\boldsymbol{u = \begin{cases} e^{i\,\omega\,(t - x/c_1)} + R\, e^{i\,\omega\,(t + x/c_1)} & \text{for } x < 0 \\ a\, e^{i\,\omega\,(t - x/c_2)} + b\, e^{i\,\omega\,(t + x/c_2)} & \text{for } 0 < x < L \\ \mathcal{T}_c\, e^{i\,\omega\,(t - x/c_3)} & \text{for } L < x \end{cases}} \tag{5.3}$$

   where $\boldsymbol{\omega > 0}$, $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{R}$, and $\boldsymbol{\mathcal{T}_c}$ are constants. Note: ω is a free constant, everything else is a function of ω.
   Important: Make sure that your calculations do not involve division by zero!

[9] Given by $\boldsymbol{c_j = \sqrt{T/\rho_j}}$, where $\boldsymbol{T}$ is the (common) string tension and $\boldsymbol{\rho_j}$ are the respective string densities. As usual, we assume a tension that is constant throughout, so that neglecting longitudinal movements is justified.

2. **Why do all the terms in (5.3) have the same time dependence? How does this follow from (5.2)?**
3. **Under which conditions is there no reflection, $R = 0$? (i.e.: impedance matching)?**
4. **Is it possible to get no transmission, $\mathcal{T}_c = 0$?**

This problem illustrates the principle behind lens coating, used in optics to prevent reflection losses in complex multi-lense devises, — *in a very simplified context.*

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6 Optical ray in a stratified media

Statement: Optical ray in a stratified media

Consider a **nearly straight and nearly horizontal light ray in a slightly stratified media.** The equation for the ray is

$$\frac{d}{d\tau}\left(\frac{n}{v}\frac{d\vec{x}}{d\tau}\right) = v\,\nabla n, \qquad (6.1)$$

where τ is some arbitrary parameter, c_0 is the speed of light in vacuum, $n = c_0/c$ is the index of refraction, and $v = |\frac{d\vec{x}}{d\tau}|$. You can find this equation in the *Lectures on Average Lagrangian*, section: *Examples*, subsection: *Wave equation and the Eikonal equation*, subsubsection: *Fermat's principle.*

Here we will assume 2-D, with $\vec{x} = (x,\, y)$ (x being the horizontal coordinate and y the vertical), with the weak stratification characterized by $n = n_0\,(1 - \epsilon\,(y - y_0))$, with $n_0 > 0$ and $0 < |\epsilon|\, L \ll 1$ (L is a length scale).
You have now the following tasks:

Task #1. Approximately compute the ray defined by the properties: it is horizontal at $x = 0$, with $y = y_0$.

Task #2. What value of ϵ yields $y = y_0 - 10\,$m for $x = 10\,$km?

For task #1. Use x as the parameter for the ray, i.e.: $y = y(x)$, and **show that** the two equations in (6.1) reduce to the single one

$$\frac{d}{dx}\frac{n}{v} = 0. \qquad (6.2)$$

To do this, first **show that** $y' = \frac{dy}{dx}$ **cannot vanish** — except for isolated points; e.g.: $x = 0$. That is: **show that** *solutions to* (6.2) *that vanish on an interval* <u>*do not*</u> *correspond to solutions to* (6.1). As a side issue, answer the question:

Task #3. Does (6.2) have solutions such that y' vanishes of an interval, but not everywhere?

Task #4. The approximate solution that you will obtain using hint #1 below is not be valid for all values of x, because it does not satisfy "y' is small" everywhere. **Where is this solution valid?**
Important: when doing task #2, **verify that you use the approximate solution within its range of validity.**

Task #5. Finally, describe real world phenomena that this calculation can help to elucidate.

Hint #1. Use that $y \approx y_0$ to solve (6.2). In particular, that $\frac{dy}{dx} = y'$ **is small.**

Hint #2. For task #3, look at what the equation reduces to when y' is small (i.e.: use the result of hint #1).

THE END.