Stress tensor short notes

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Abstract

The notion of stress tensor is introduced to characterize the internal forces in continuum media.

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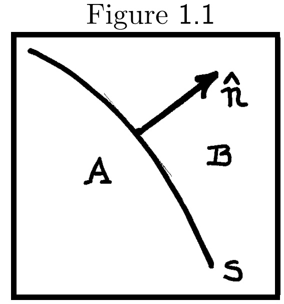
1 Forces in a continuous media

Consider some substance/material under stress. Introduce an inertial, cartesian, coordinate system, $\vec{\boldsymbol{x}}$. Let $\boldsymbol{\mathcal{S}}$ be some surface within the material. For any surface element $\mathbf{d}\vec{\boldsymbol{S}}$, with unit normal $\hat{\boldsymbol{n}}$ (see the figure and item **a2**), let:

$$\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\hat{\boldsymbol{n}},\, \vec{\boldsymbol{x}}) = \text{ force per unit area across the surface, by the material on side B, on the material on side A.} \tag{1.1}$$

Figure 1.1



Note that $\vec{\boldsymbol{F}}$ = **linear momentum flux.** Hence, by the same argument as in the problem "*The flux for a conserved quantity must be a vector*", we conclude that:

$\vec{\boldsymbol{F}}$ **is given by a tensor;** that is: † $\quad \vec{\boldsymbol{F}} = \boldsymbol{\tau} \cdot \hat{\boldsymbol{n}}$ or, by components, $\boldsymbol{F_n} = \boldsymbol{\tau_{nm}}\, \boldsymbol{n_m}$. (1.2)

In elasticity, τ is called the *Cauchy stress tensor.*

† See the notation subsection, §1.6, at the end.

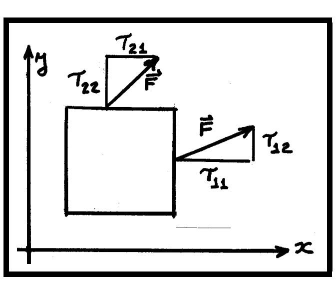


Figure 1.2: The meaning of $\boldsymbol{\tau}$ in 2-D

Forces on an "elementary" square, lined up with the coordinate axis, by the media outside. For the edges parallel to the y-axis, the components of the forces are $\boldsymbol{\tau_{11}}$ and $\boldsymbol{\tau_{12}}$. For the edges parallel to the x-axis, the components of the forces are $\boldsymbol{\tau_{21}}$ and $\boldsymbol{\tau_{22}}$.

Examples: In elasticity $\boldsymbol{\tau}$ is a function of the strain tensor $\boldsymbol{\sigma}$; for Newtonian fluids $\boldsymbol{\tau}$ is a function of the pressure and the gradients of the fluid velocity.

1.1 Assumptions

Here are some important assumptions.

a1. Continuum hypothesis. The media can be characterized in terms of various fields defined in space and time. As functions of $(\vec{x},\, t)$ these fields are reasonably well behaved: they are smooth almost everywhere, and their singularities (if any) do not give rise to physical contradictions (e.g.: infinite energy, infinite accelerations, and so on). The main theoretical objective is then to provide equations that uniquely characterize the evolution of these fields.

a2. Cauchy's postulate. The stress tensor, as introduced above, depends on the assumption that the force between the sides of an arbitrary surface across the material is a function of the surface normal only — this is *Cauchy's postulate*. Note that this assumption is not always valid; for example: at interfaces between different fluids other forces arise, such as surface tension and Marangoni effects.

1.2 The stress tensor is symmetric

Here we show that (as a consequence of the continuum hypothesis **a1**) $\boldsymbol{\tau}$ is symmetric: $\tau_{nm} = \tau_{mn}$.

Proof. Consider a sphere of radius $R > 0$ centered at some arbitrary point, $\vec{x}_0$, in the media. Denote this sphere by Ω, and its boundary by $\partial\Omega$. The torque on the sphere is then †

$$\begin{aligned} \vec{T} &= -\int_{\partial\Omega} (\vec{x} - \vec{x}_0) \times (\boldsymbol{\tau} \cdot \hat{n})\, \mathrm{d}A = -\int_{\Omega} \mathrm{div}\left((\vec{x} - \vec{x}_0) \times \boldsymbol{\tau}\right)\, \mathrm{d}V \\ &= \underbrace{-\int_{\Omega} (\vec{x} - \vec{x}_0) \times \mathrm{div}(\boldsymbol{\tau})\, \mathrm{d}V}_{I_1} - \underbrace{\int_{\Omega} \{\varepsilon_{npq}\, \tau_{qp}\}\, \mathrm{d}V}_{I_2} \end{aligned} \tag{1.3}$$

where $\hat{n}$ is the outside unit normal to the sphere, $\mathrm{d}A$ is the area differential, and $\mathrm{d}V$ is the volume differential.

† **Details.**

Equality #1. Follows from the definition of torque and the definition of $\boldsymbol{\tau}$.

Equality #2. $(\vec{z} \times (\boldsymbol{\tau} \cdot \hat{n}))_n = \varepsilon_{npq}\, z_p\, \tau_{qm}\, n_m$. Then use Gauss' theorem.

Equality #3. $(\mathrm{div}\,(\vec{z} \times \boldsymbol{\tau}))_n = (\varepsilon_{npq}\, z_p\, \tau_{qm})_{x_m} = \varepsilon_{npq}\, \delta_{pm}\, \tau_{qm} + \varepsilon_{npq}\, z_p\, (\tau_{qm})_{x_m}$
$= \varepsilon_{npq}\, \tau_{qp} + (\vec{z} \times \mathrm{div}(\boldsymbol{\tau}))_n$.

However, for Ω fixed

$$\underbrace{\frac{\mathrm{d}}{\mathrm{dt}} \int_{\Omega} \rho\, \vec{v} \times (\vec{x} - \vec{x}_0)\, \mathrm{d}V}_{I_3} = \vec{T} + \underbrace{\int_{\Omega} \vec{f} \times (\vec{x} - \vec{x}_0)\, \mathrm{d}V}_{I_4} \tag{1.4}$$

where ρ is the mass density, $\vec{v}$ is the substance flow velocity, and $\vec{f}$ are the body forces (if any). Now, at any non-singular point $\vec{x}_0$, as $R \to 0$, the various integrals above satisfy $I_j = O(R^4)$, **except** for I_2, which behaves like $O(R^3)$ if $\{\varepsilon_{npq}\, \tau_{qp}\} \neq 0$. It follows that it must be $\{\varepsilon_{npq}\, \tau_{qp}\} = 0$, which is equivalent to $\boldsymbol{\tau}$ being symmetric. ♣

1.3 Transformation properties

Consider two cartesian coordinate systems, $\vec{x}_a$ and $\vec{x}_b$, related by an **orthogonal transformation** T. Specifically: $\vec{x}_b = T\,\vec{x}_a$. Then

$$\tau_b = T\,\tau_a\,T^T. \tag{1.5}$$

Proof. From (1.2), $\vec{F}_a = \tau_a\,\hat{n}_a$ and $\vec{F}_b = \tau_b\,\hat{n}_b$. However, it must also be $\vec{F}_b = T\,\vec{F}_a$ and $\hat{n}_b = T\,\hat{n}_a$. Thus $\tau_b\,T\,\hat{n}_a = \tau_b\,\hat{n}_b = T\,\vec{F}_a = T\,\tau_a\,\hat{n}_a$. Since this must apply for all $\hat{n}_a$, it must be $\tau_b\,T = T\,\tau_a$. ♣

1.4 Pressure-shear decomposition

Let $p = -\frac{1}{3}\,\mathrm{Tr}(\tau) = -\frac{1}{3}\,\tau_{nn}$. Then

$$\tau = -p\,I + \tau^*, \tag{1.6}$$

where τ^* is symmetric and trace-less, †, and p is the **pressure.**

† Hence the sum of the three principal forces by τ^* vanishes — see §1.5.

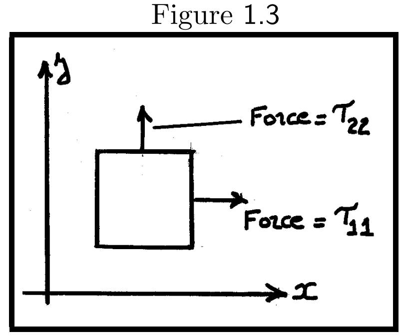
The pressure is the isentropic part of the stress tensor. If τ^* vanishes, the stress is normal to any surface, and independent of the direction. The response to pressure only by an isentropic media is a volume change. Vice-versa, in isentropic materials (elastic or fluid) a volume change causes a pressure. ‡

‡ In dissipative media a pressure can be caused by the rate of change of the volume (opposing the change). For example, in Newtonian fluids there can be a pressure due to viscous effects: $p = \kappa\,\mathrm{div}(\vec{v})$, where κ is the bulk viscosity and $\vec{v}$ is the flow velocity.

On the other hand, if $p \equiv 0$, the response of an isentropic material is an equi-volume shape change. In Newtonian fluids τ^* is produced by shear (proportional to the shear viscosity), while in elastic media undergoing small local deformations it results from the shear component of the deformations.

1.5 Principal axes

Figure 1.3



Because τ is symmetric, in an infinitesimal neighborhood of any point a cartesian system of coordinates exist where τ is diagonal — select the appropriate T in (1.5). These are the **principal axes.**

In the principal axes coordinates (see the 2-D example in the picture), the forces are normal to any surface parallel to a coordinate plane. Then, for an "elementary cube" aligned with the axes only squeeze and pull happens; there is no shear along the facets of this cube.

1.6 Notation

Here we summarize some of the conventions and notations that we use.

n1. For the purpose of these notes, there are only two types of tensors: rank-one, or (column) vectors, and rank-two or (square) matrices. A dot between two tensors is used to mean "contraction" of the last index of the left factor with the first index of the second factor. Thus for vectors $\vec{a}\cdot\vec{b} = \sum_n a_n\,b_n$ is the scalar product, while for matrices (either $(A\cdot B)_{nm} = \sum_j A_{nj}\,B_{jm}$ or $(A\cdot\vec{b})_n = \sum_j A_{nj}\,b_j$) it is matrix multiplication. However, note that $\vec{b}\cdot A = \vec{b}^T A$ since $(\vec{b}\cdot A)_m = \sum_m b_n\,A_{nm}$.

n2. The **divergence** operator is defined either by $\mathrm{div}(\vec{f}) = \sum_n (f_n)_{x_n}$ for a vector (the result is a scalar) or $\mathrm{div}(\tau) = \{\sum_m (\tau_{nm})_{x_m}\}$ for a matrix (the result is a vector).

n3. We will often use the **repeated index summation convention** to simplify the notation. For example, using this convention $\text{div}(\boldsymbol{\tau}) = (\boldsymbol{\tau}_{nm})_{x_m}$, and the summation symbol over m is not needed (because m is repeated).

n4. $\boldsymbol{\nabla}$ is the vector operator with components $\boldsymbol{\partial}_{x_n}$.

n5. $\boldsymbol{\tau}^T$ is the transpose of $\boldsymbol{\tau}$, defined by $(\boldsymbol{\tau}^T)_{nm} = \boldsymbol{\tau}_{mn}$.

n6. $\boldsymbol{\delta}_{nm}$ is the Kronecker delta: $\boldsymbol{\delta}_{nm} = \mathbf{0}$ if $\boldsymbol{n} \neq \boldsymbol{m}$ and $\boldsymbol{\delta}_{nm} = \mathbf{1}$ if $\boldsymbol{n} = \boldsymbol{m}$.

n7. $\boldsymbol{\varepsilon}_{npq}$ is the permutation index: $\boldsymbol{\varepsilon}_{npq} = \mathbf{0}$ if there are repeated indexes, $\boldsymbol{\varepsilon}_{npq} = \mathbf{1}$ if $(\boldsymbol{n},\ \boldsymbol{p},\ \boldsymbol{q})$ has the same ordering as $(\mathbf{1},\ \mathbf{2},\ \mathbf{3})$, and $\boldsymbol{\varepsilon}_{npq} = -\mathbf{1}$ if $(\boldsymbol{n},\ \boldsymbol{p},\ \boldsymbol{q})$ has the reverse ordering to $(\mathbf{1},\ \mathbf{2},\ \mathbf{3})$.

n8. The vector product is defined by: For two vectors $\vec{x}$ and $\vec{\boldsymbol{y}}$ it is the vector $(\vec{\boldsymbol{x}} \times \vec{\boldsymbol{y}})_n = \boldsymbol{\varepsilon}_{mpq}\, \boldsymbol{x}_p\, \boldsymbol{y}_q$.
For a vector $\vec{\boldsymbol{x}}$ and a matrix $\boldsymbol{A}$ it is a matrix;
either $(\boldsymbol{A} \times \vec{\boldsymbol{x}})_{nm} = \boldsymbol{\varepsilon}_{mpq} \boldsymbol{A}_{np}\, \boldsymbol{x}_q$ or $(\vec{\boldsymbol{x}} \times \boldsymbol{A})_{nm} = \boldsymbol{\varepsilon}_{npq}\, \boldsymbol{x}_p\, \boldsymbol{A}_{qm}$.

The End.