

Continuum Media and Elasticity

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Contents

1 Elasticity: Set up and Forces	2
Definition of an elastic solid	2
Definition of stress tensor	2
Definition of non pre-stressed elastic solid	3
Description of an elastic solid: Lagrangian and Eulerian coordinates	3
Displacement field	3
Definition of pure shear and simple shear	3
1.1 Summary of assumptions	3
1.2 Local description: infinitesimal neighborhood	4
Deformation decomposition: rotation, pure shear, and compression or expansion	4
Example: small deformations	5
1.2.1 The strain tensor and the elastic energy	5
Stress-Strain tensor relationship	5
1.2.2 Tracking angles and lengths (scalar products and the strain tensor)	6
1.2.3 Transformation properties of the strain tensor and isotropic media	6
1.2.4 Other fields used	6
2 Problem: The stress tensor is symmetric	7
Use the conservation of angular momentum to show that the stress tensor is symmetric	7
2.1 Statement: The stress tensor is symmetric	7
3 Problem: Uniform strain elasticity solution	8
Construct a special solution of the elasticity equations, corresponding to a uniform, but arbitrary, strain field	8
3.1 Statement: Uniform strain elasticity solution	8
4 Elasticity: Energy considerations	9
4.1 Potential energy	9
Potential energy in the small deformation limit	9
4.2 The relationship between V and τ	10
Stress tensor in terms of the elastic deformation energy	10
4.2.1 The general isotropic media case	11
5 Elasticity: The 1-D case	11
5.1 Equations of motion	12
5.1.1 Other conservation laws	12
5.1.2 Eulerian coordinates	13
6 Problem: Conservation laws in elasticity	13
(i) Derive the equations for a continuum. (ii) Calculate the evolution along particle paths of the internal energy. (iii) Show: angular momentum conserved if and only if τ is symmetric	13
6.1 Statement: Conservation laws in elasticity	13

7 Elasticity: The small deformations limit	14
7.1 Hooke's law	14
Stress-strain relationship for isotropic media	15
Lamé constants and the shear modulus	15
Colinearity of the stress and the strain	15
Should the Lamé constants be positive?	15
Elastic deformation energy	15
7.1.1 The other constants	16
Bulk modulus	16
Pressure and relationship to volume change	16
Shear modulus	16
Young's modulus	16
Poisson ratio	16
7.1.2 The consequences of symmetry	17
8 Problem: Energy in linear elasticity	17
Directly verify conservation of energy in linear elasticity	17
8.1 Statement: Energy in linear elasticity	17
9 Euler-Lagrange equations (variational formulation)	18
The Piola-Kirchhoff stress tensors	18
9.1 Conservation of angular momentum	19
9.2 Transformation to Eulerian coordinates	19
9.2.1 Forces and stress tensors	19
9.3 A few useful formulas	20

1 Elasticity: Set up and Forces

We begin by precisely defining the assumptions/idealizations that characterize elastic solids.

Definition 1.1 An **elastic solid** is characterized by: *each infinitesimal element in the solid has an equilibrium shape, to which it returns if all the external forces on the element are relaxed. To be more precise: deformations of the equilibrium shape generate forces that oppose the deformations, and the forces depend on the deformations only.* † The forces are short range (infinitesimal in the continuum limit) and can be described by a **stress tensor**.

† For example, the forces do not depend on the rate of deformation, which would cause dissipation — see remark 1.3. ♣

Definition 1.2 The **stress tensor** $\tau = \{\tau_{ij}\}$ is defined as follows: Consider an arbitrary surface \mathcal{S} within the solid, dividing the solid into two parts, say #1 and #2. Let \hat{n} be the unit normal to \mathcal{S} , pointing into the region #2. Then the force per unit area, exerted by region #2 on region #1 across the surface, is given by $\vec{f}^a = \tau \cdot \hat{n}$. ♣

As defined, τ is called the *Cauchy stress tensor* (see item **a5** in §1.1).

Note 1.2a Component by component, $\vec{f}^a = \tau \cdot \hat{n}$ means: $f_i^a = \sum_j \tau_{ij} n_j$. Similarly: $\text{div}(\tau) = \left\{ \sum_j (\tau_{ij})_{x_j} \right\}$.

Note 1.2b We will often use the **repeated index summation convention** to simplify the notation. For example, using this convention $\text{div}(\tau) = (\tau_{ij})_{x_j}$, and the summation symbol over j is not needed (because j is repeated).

Note 1.2c Starting from *Cauchy's postulate* (see item **a5** in §1.1), that the forces are characterized by a stress tensor can be shown by an argument entirely analogous to the one used in the problem: *The flux for a conserved quantity must be a vector.*

Note 1.2d It is **important** to point out that: in this definition normals and areas are in the “physical space” (Eulerian) coordinates \vec{x} , **not** the reference frame (Lagrangian) coordinates \vec{s} we introduce below (1.1).

Remark 1.1 *The stress tensor is symmetric.* See §2. ♣

Definition 1.3 A **non pre-stressed elastic solid** is an elastic solid such that *the whole solid* (not just each infinitesimal element) possesses an equilibrium shape, where the stress tensor vanishes.

In these notes we deal with non pre-stressed elastic solids only.

Pre-stressed elastic solids are quite common; it is very hard to obtain truly non pre-stressed materials. Most elastic objects, when all the external forces are removed, relax to an equilibrium state where some stresses remain: each element could relax to a zero stress shape, but their arrangement in space does not allow this simultaneously. *However, as long as the deformation scales are much larger than the scale of the “residual” stresses, the non pre-stressed approximation is reasonable.* ♣

Given a non pre-stressed elastic solid, introduce an inertial, cartesian, coordinate system and, for each point \vec{x} in the solid, let:

$$\vec{s} = \text{coordinates of the point when the solid is at rest and in its equilibrium shape.} \quad (1.1)$$

Then \vec{s} provides a system of *Lagrangian coordinates* for the solid — i.e.: a label for each mass point. The *solid is completely described*, at any time, by the function

$$\vec{x} = \vec{X}(\vec{s}, t), \quad \text{or its inverse} \quad \vec{s} = \vec{S}(\vec{x}, t), \quad (1.2)$$

relating the current and equilibrium positions of each mass point. We can also write

$$\vec{x} = \vec{s} + \vec{u} \quad \text{or} \quad \vec{s} = \vec{x} - \vec{u}, \quad \text{where} \quad \vec{u} = \vec{u}(\vec{x}, t) = \vec{u}(\vec{s}, t) = \text{displacement field.} \quad (1.3)$$

The *objective is now to describe the equations that govern the time evolution of \vec{X} .*

Remark 1.2 *The definition of \vec{s} in (1.1) is not unique. Given some Lagrangian solid coordinates \vec{s} , for any $R =$ constant rotation matrix \dagger and $\vec{s}_0 =$ constant vector, $\vec{s}_* = R\vec{s} + \vec{s}_0$ are also acceptable Lagrangian coordinates. Here we **assume that some choice has been made.***

\dagger A rotation is characterized by $R^{-1} = R^T$ and $\det(R) = 1$. ♣

Definition 1.4 Pure shear occurs when there is a cartesian system of axes such that the deformation is either elongation, or contraction, along each of the axis; while the volume is preserved. *Example:* a rubber ball deformed by pressing along opposite ends of a ball diameter. Note that *there is no rotation associated with pure shear.*

Simple shear is a deformation in which some family parallel planes in the material remain parallel and maintain a constant distance, while translating relative to each other. *Example:* a rod under torsion along its axis. In general, *simple shear involves rotation.* ♣

1.1 Summary of assumptions

Here is a list of important assumptions that are used throughout these notes, often implicitly. The consequences of these assumptions are far-reaching, but listing them every time they are used would lead to a rather awkward presentation. These *apply to non pre-stressed, elastic solids.*

- a1.** Some of the formulas apply for isotropic media only. We will identify these explicitly.
- a2.** We neglect dissipation — see remark 1.3.
- a3.** Work done on (or by) the media is stored in the media deformations as elastic energy.

- a4.** The properties of the media (e.g.: forces/stresses, elastic energy) are local, and depend on the local deformations only. In particular, these properties do not depend on how the current state was achieved. That is: no path (history) dependence.

Mathematically: the media properties are point functions of the strain tensor σ , defined in §1.2.1. For example: $\tau(\vec{x}, t)$ depends only on $\sigma(\vec{x}, t)$, and media “constants” that may be functions of \vec{s} . Nothing else.

- a5.** The stress tensor, as introduced it in definition 1.1, depends on the assumption that the force between the sides of an arbitrary surface across the solid is a function of the surface normal only — this is called the *Cauchy’s postulate*. Generally this is only valid for relatively small deformations. For larger deformations dependences beyond the local normal \hat{n} might also play a role — e.g.: the curvature of the surface may matter.
- a6.** We will only consider solids such that there is a “deformation path” that connects the solid state to its equilibrium configuration in a continuous way. This is a stronger restriction than non-prestressed; i.e.: not only does an equilibrium shape exists, but it must be reachable through a continuous deformation process.

Remark 1.3 On dissipation and plasticity. *When a plastic deformation occurs, the media’s equilibrium state changes. Then the assumption in a4 is violated, and memory of the past deformations is required to reconstruct the media properties. In this case equations for how the equilibrium state evolves (as well as the media parameters — e.g.: elastic stiffness) are needed. † There may be other ways in which a4 is violated.*

† The theory then becomes considerably more complicated than non-linear elasticity, which is already quite challenging.

When a plastic deformation occurs, energy is lost. However, this is not the only way in which dissipation arises, as the example of a damped mass-spring system illustrates. This type of dissipation does not invalidate the assumptions in items a3–a4, provided that it can be described in terms of “dissipative forces”. Then, when work is done on (or by) the media some of the work is done against the dissipative forces, and the elastic energy only changes due to the net work. But the basic framework of the theory survives. †

† Important proviso: The framework survives, but the mathematical properties of the theory can change dramatically.

We will deal with neither of these issues here, nor item a5.



1.2 Local description: infinitesimal neighborhood

Consider an infinitesimal neighborhood of some fixed point, and write

$$\vec{x} = \vec{x}_0 + \delta\vec{x} \quad \text{and} \quad \vec{s} = \vec{s}_0 + \delta\vec{s}, \quad (1.4)$$

where $\vec{x}_0 = \vec{X}(\vec{s}_0, t)$. Then

$$\delta\vec{x} = \mathbf{G} \delta\vec{s} \quad \text{and} \quad \delta\vec{s} = \mathbf{H} \delta\vec{x}, \quad (1.5)$$

where \mathbf{G} and \mathbf{H} are the matrices defined by $G_{ij} = \frac{\partial x_i}{\partial s_j}$ and $H_{ij} = \frac{\partial s_i}{\partial x_j}$. Note that: $\mathbf{G} = \mathbf{H}^{-1}$.

We can write¹

$$\mathbf{G} = d \mathbf{G}_r \mathbf{G}_s \quad (\text{thus: } \delta\vec{x} = d \mathbf{G}_r \mathbf{G}_s \delta\vec{s}), \quad (1.6)$$

where $d > 0$ is a scalar, \mathbf{G}_r is a **rotation matrix** (orthogonal with determinant one), and \mathbf{G}_s is a **symmetric matrix with positive eigenvalues**, $\mathbf{G}_s > 0$, and **determinant one**. This factorization of \mathbf{G} shows that **all the local deformations of the solid can be reversed by first applying a rotation (\mathbf{G}_r^{-1}), followed by pure shear (\mathbf{G}_s^{-1}), and either expansion (if $d < 1$) or contraction (if $d > 1$)**. Since we can also write $\mathbf{G} = d \tilde{\mathbf{G}}_s \tilde{\mathbf{G}}_r$, the order (pure shear first) of these operations can be changed. The expansion/contraction can be done first, second, or last.

The factorization in (1.6) follows from the fact that any matrix can be written as the product of a rotation matrix times a symmetric matrix (polar decomposition), and item **a6** in §1.1:

$$\textit{the solid deformations are the result of a continuous process starting from equilibrium.} \quad (1.7)$$

¹ Note that a similar decomposition exists for \mathbf{H} .

The proof is as follows: From the polar decomposition we can write $G = G_r S$, where S is symmetric. But G is not singular, hence $G = d G_r G_s$, where $d = (\det(S))^{1/3}$, and $G_s = d^{-1} S$. This formula should apply at every step of the deformation process from equilibrium — hence neither d , nor any of the eigenvalues of G_s can cross zero. But, at equilibrium: $d = 1$ and $G_r = G_s = \text{identity}$. Hence $d > 0$ and $G_s > 0$. ♣

Example 1.1 (Small deformations). Use (1.3) to write $H = \mathbf{1} - D$, where $D_{ij} = \frac{\partial u_i}{\partial x_j}$. (1.8)

Now assume small deformations: $D = O(\epsilon)$, where $0 < \epsilon \ll 1$. Then $G = \mathbf{1} + D + O(\epsilon^2)$

and $G = 1 + \alpha + D_a + D_s + O(\epsilon^2) = (1 + \alpha)(1 + D_a)(1 + D_s) + O(\epsilon^2) = (1 + \alpha)e^{D_a}e^{D_s} + O(\epsilon^2)$, (1.9)

where $\alpha = \text{Tr}(D)$, $D_a = \frac{1}{2}(D - D^T)$ is the anti-symmetric part of D , and $D_s = -\alpha + \frac{1}{2}(D + D^T)$ is the trace-less symmetric part of D . Thus

$$d = (1 + \alpha) + O(\epsilon^2), \quad G_r = e^{D_a} + O(\epsilon^2), \quad \text{and} \quad G_s = e^{D_s} + O(\epsilon^2), \quad (1.10)$$

which follows by comparing (1.6) and (1.9). ♣

1.2.1 The strain tensor and the elastic energy

Let V denote the elastic energy per unit mass — see item **a3** in §1.1. This **energy should only be a function of the shape changes in the solid**. Now, the information about the deformations (shape changes) in the solid is encoded in the dG_s factor of the representation in (1.6). The rotations, encoded in G_r , affect neither the geometry† of the forces (stresses) generated when the elastic solid is deformed, nor the elastic energy accumulated. Hence *the elastic energy should be a function of dG_s only*, and the same should apply to the geometry of the stresses.

† The force “diagram” should just rotate with the media, without any change in its geometry.

At this point it is convenient to introduce the **strain tensor**

$$\sigma = \frac{1}{2}(G^T G - \mathbf{1}), \quad \text{i.e.:} \quad \sigma_{ij} = \frac{1}{2} \left(\frac{\partial x_\ell}{\partial s_i} \frac{\partial x_\ell}{\partial s_j} - \delta_{ij} \right), \quad (1.11)$$

where δ_{ij} is the *Kronecker delta* and we use the repeated index summation convention. Note that σ is **both a-dimensional and symmetric**. Further: $\sigma = \frac{1}{2}(d^2 G_s^2 - \mathbf{1})$. Hence, since $d > 0$ and $G_s = G_s^T > 0$, it should be clear that σ encodes † the same information as dG_s .

† dG_s is the (unique) symmetric and positive definite solution to $(dG_s)^2 = G^T G = 2\sigma + \mathbf{1}$. Then $d = (\det(dG_s))^{1/3}$.

Hence, **for an elastic non pre-stressed solid**

$$V = V(\sigma), \quad \tau = G \tilde{N}(\sigma) G^T, \quad \text{and} \quad \tilde{N}(\mathbf{0}) = \mathbf{0}, \quad (1.12)$$

where $\tilde{N} = \tilde{N}(\sigma)$ is some tensor, symmetric because τ is symmetric — see §2.

Remark 1.4 Equation (1.12) applies in the case of an homogeneous elastic solid.

If the solid is not homogeneous, then $\tilde{N} = \tilde{N}(\vec{s}, \sigma)$, with $\tilde{N}(\vec{s}, \mathbf{0}) \equiv \mathbf{0}$.

Of course, $G = G(\vec{s}, t)$ and $\sigma = \sigma(\vec{s}, t)$, as follows from (1.5) and (1.11). ♣

The definition of σ in (1.11), and (1.12), require some clarifications:

A. As mentioned earlier (below (1.10)), we can also write $G = d \tilde{G}_s \tilde{G}_r$, where $\tilde{G}_s > 0$ is symmetric with $\det \tilde{G}_s = 1$, and \tilde{G}_r is a rotation. So, why not use $G G^T = d^2 \tilde{G}_s^2$ to define the strain tensor? The reason is that $G^T G$ cancels out rotations in the physical space (which should not affect the geometry of the forces), while $G G^T$ cancels out rotations in the reference space (which might affect the forces: see §1.2.3). Specifically:

(i) Applying a rotation, R , in physical space should not change the elastic energy, and just rotate the forces. Thus V should remain invariant, while the forces should transform by $\vec{f} \rightarrow R \vec{f}$.

(ii) A rotation, R , in physical space changes $G \rightarrow RG$, and normals by $\hat{n} \rightarrow R\hat{n}$. Thus (1.12) behaves precisely as required: V does not change, while the forces produced by τ , $\vec{f}^a = \tau \cdot \hat{n}$, rotate by R .

B. Why not use H to define σ ? Well, since $H = G^{-1}$, it is fully equivalent to G , and one could use it as well — the appropriate object is then HH^T . Since here we will think of $\vec{X}(\vec{s}, t)$ as the variable to be solved for, G is more convenient.

C. As the arguments in item **A** show, the pre and post-factors G and G^T in (1.12) are there to ensure τ behaves correctly under rotations. This could also be achieved by, for example, $\dagger \tau = H^T N_*(\sigma) H$. Then N_* and \tilde{N} are related by $N_* = G^T G \tilde{N} G^T G = (2\sigma + 1) \tilde{N} (2\sigma + 1)$. Which form to use is a matter of convenience.

\dagger Mixed forms, such as $\tau = G N_{\#} H$ are not good because they require a non-symmetric $N_{\#}$.

D. Finally: (i) *Why do we subtract 1 from $G^T G$ to define σ ?* So that σ characterizes departures from equilibrium. (ii) *What is the purpose of the pre-factor $\frac{1}{2}$ in (1.11)?* So that, in the limit of infinitesimal deformations, the formula yields the standard one in linear elasticity. (iii) *Why is the prefactor positive and not negative?* Because then stretching/compression correspond to positive/negative eigenvalues of σ — this is just convention.

1.2.2 Tracking angles and lengths (scalar products and the strain tensor)

Another way to see why σ is the correct object is as follows: the amount of deformation in an elastic solid can be determined if we know how the distances and angles change. This is equivalent to knowing how the scalar product changes. It is easy to see that

$$\langle \delta \vec{x}_1, \delta \vec{x}_2 \rangle = \langle \delta \vec{s}_1, (2\sigma + 1) \delta \vec{s}_2 \rangle, \quad \text{since } \delta \vec{x} = G \delta \vec{s}. \quad (1.13)$$

The strain tensor encodes the information on how distances and angles change by the deformation of the solid from equilibrium. This argument does not differentiate $G^T G$ from $G G^T$. To do so an extra restriction must be imposed: *invariance under rotation of the coordinates in the physical frame* — see item **A** in §1.2.1

1.2.3 Transformation properties of the strain tensor and isotropic media

By construction **the strain tensor $\sigma = \frac{1}{2}(G^T G - 1)$ is invariant under rotations in the Eulerian frame, $\delta \vec{x} \rightarrow R_e \delta \vec{x}$** (here R_e is a rotation), because these correspond to $G \rightarrow R_e G$. However,

σ is not invariant under rotations in the Lagrangian frame,

$\delta \vec{s} \rightarrow R_l \delta \vec{s}$. These correspond to $G \rightarrow G R_l^T$, \dagger so that
$$\sigma \rightarrow R_l \sigma R_l^T. \quad (1.14)$$

\dagger Proof. Let $\delta \vec{s}_* = R_l \vec{s}$ be the rotated coordinates. Then $\delta \vec{x} = G \vec{s} = G R_l^T \delta \vec{s}_*$.

A similar argument shows that $\delta \vec{x} \rightarrow R_e \delta \vec{x}$ yields $G \rightarrow R_e G$.

It then follows that

$$V \rightarrow V(R_l \sigma R_l^T) \text{ and } \tau \rightarrow G R_l^T \tilde{N}(R_l \sigma R_l^T) R_l G^T. \quad (1.15)$$

But, *what is the physical meaning of a rotation in the reference frame?* It means that the same solid “shape” has been achieved by deforming the at-equilibrium solid along different axes. Thus *the resulting stresses and elastic energy need not be the same*, unless the solid is isotropic. We conclude that:

$$\mathbf{A \ solid \ is \ isotropic} \iff V(\sigma) = V(R_l \sigma R_l^T) \text{ and } \tilde{N}(\sigma) = R_l^T \tilde{N}(R_l \sigma R_l^T) R_l. \quad (1.16)$$

for any rotation matrix R_l .

1.2.4 Other fields used

$\rho_0 = \rho_0(\vec{s})$	mass density at equilibrium.
$\rho = \rho(\vec{x}, t) = \rho_0 \det(H)$	mass density (mass per unit volume).
$\vec{v} = \vec{v}(\vec{x}, t) = \mathbf{X}_t(\vec{s}, t)$	mass flow velocity = $\frac{d\vec{x}}{dt}$ keeping \vec{s} constant.
$\vec{f} = \vec{f}(\vec{x}, t)$	body force per unit mass.

2 Problem: The stress tensor is symmetric

2.1 Statement: The stress tensor is symmetric

Consider a continuous media filling some region of space \mathcal{R} , such that the inter-media forces in it are characterized by a **stress tensor** $\tau = \{\tau_{ij}\}$, defined at every point $\vec{x} \in \mathcal{R}$.

Note: the stress tensor is defined as follows: *Consider an arbitrary surface \mathcal{S} within \mathcal{R} , dividing the region into two parts, say #1 and #2. Let \hat{n} be the unit normal to \mathcal{S} , pointing into the region #2. Then the force per unit area, exerted by region #2 on region #1 across the surface, is given by $\vec{f} = \tau \cdot \hat{n}$.*

Under the assumptions listed below, **show that:**

$$\text{The stress tensor } \tau \text{ is symmetric.} \quad (2.1)$$

Assumptions

- a1.** The following fields can be defined: $\rho = \rho(\vec{x}, t)$ = mass density (mass per unit volume), $\vec{v} = \vec{v}(\vec{x}, t)$ = mass flow velocity, $\vec{f} = \vec{f}(\vec{x}, t)$ = body force per unit mass, and (of course) the stress tensor $\tau = \tau(\vec{x}, t)$.
- a2.** The **only forces present** in the system are those associated with \vec{f} and τ .
- a3.** *Mathematical assumptions:* \mathcal{R} is open, and all the fields in item **a1** are continuously differentiable.

Remark 2.5 For a complete description of the system more fields than the ones introduced in item **a1** are needed (e.g.: pressure, temperature, chemical composition, internal energy, etc.). In addition, *equations of state* that relate the various fields are needed; examples: (i) For inert gases the internal energy is given as a function of the density and the pressure. (ii) For non pre-stressed elastic solids, a map to a reference (equilibrium) frame \vec{s} is introduced, $\vec{x} = \vec{X}(\vec{s}, t)$. Then τ is given in terms of the strain tensor $\sigma = \frac{1}{2}(G^T G - 1)$, where $G = \{G_{ij} = \frac{\partial x_i}{\partial s_j}\}$. *However, here we will not be concerned with the full set of governing equations for the media.* ♣

Hints

- h1.** Let \vec{x}_0 be an arbitrary point in \mathcal{R} , and let \mathcal{B}_h be a small² ball of radius h , $0 < h \ll 1$, centered at \vec{x}_0 . Then write the equation for the conservation of the angular momentum. That is:

$$\frac{d}{dt} \vec{A} = \vec{T}_s + \vec{T}_b + \vec{F}_s, \quad (2.2)$$
 where: (a) \vec{A} is the angular momentum contained within \mathcal{B}_h , (b) \vec{T}_s is the torque provided by the forces from the media outside \mathcal{B}_h , applied along the boundary $\partial\mathcal{B}_h$ of the ball, (c) \vec{T}_b is the torque provided by the body forces, and (d) \vec{F}_s is the flow of angular momentum across the boundary $\partial\mathcal{B}_h$ (advection). Note: *define the angular momentum, and torque, relative to the ball's center.*

Recall that the angular momentum of a particle is $m \vec{V} \times \vec{r}$, where m is the mass, \vec{V} is the velocity, and \vec{r} is the position vector. Similarly, the torque is $\vec{F} \times \vec{r}$, where \vec{F} is the applied force. Thus, in the case here, the angular momentum density is $\rho \vec{v} \times (\vec{x} - \vec{x}_0)$, the applied torque per unit volume is $\rho \vec{f} \times (\vec{x} - \vec{x}_0)$, and so on. In particular, you should be able to write the torque density applied along $\partial\mathcal{B}_h$ in terms of the stress tensor.

Each of the terms in (2.2) is thus given by an integral (volume or surface) of the corresponding density.

- h2.** Estimate the size of the terms in (2.2) as $h \rightarrow 0$, and use this to conclude that $\vec{T}_s + \vec{F}_s$ must satisfy $\vec{T}_s + \vec{F}_s = O(h^4)$. Then use Gauss' theorem to rewrite $\vec{T}_s + \vec{F}_s$ as a volume integral of the divergence of a rank two tensor. *Examination of this divergence should now allow you to conclude that $\vec{T}_s + \vec{F}_s = O(h^4)$ requires that τ be symmetric.*
- h3.** *Some mathematical facts that you will need.*

Let Ω be some region, with boundary $\partial\Omega$ and external unit normal vector $\hat{n} = \{n_j\}$. Let $a = \{a_{ij}\}$ be a rank two tensor field, continuously differentiable and defined in Ω . Then

$$\int_{\partial\Omega} a \cdot \hat{n} dS = \int_{\Omega} \text{div}(a) dV, \quad (2.3)$$
 where $a \cdot \hat{n} = \{a_{ij} n_j\}$, $\text{div}(a) = \{(a_{ij})_{x_j}\}$, and **we use the repeated index summation convention.** ♣

Let ε_{ijk} be defined as follows: (i) $\varepsilon_{ijk} = \mathbf{0}$ if there are repeated indexes. (ii) $\varepsilon_{ijk} = \mathbf{1}$ if indexes are in one of these orders: (1, 2, 3), (2, 3, 1), or (3, 1, 2). (iii) $\varepsilon_{ijk} = -\mathbf{1}$ otherwise.

Then the vector product can be written as

$$\vec{p} \times \vec{q} = \varepsilon_{ijk} q_j p_k. \quad (2.4)$$

² Small enough to be contained in \mathcal{R} .

3 Problem: Uniform strain elasticity solution

3.1 Statement: Uniform strain elasticity solution

Consider an homogeneous elastic solid, with equilibrium density $\rho_0 > 0$.

Let $\mathbf{G} = \mathbf{G}(t)$ be a 3×3 matrix valued smooth function of time, such that:

$$\mathbf{G} = \mathbf{G}^T > \mathbf{0}. \quad (3.1)$$

That is: \mathbf{G} is symmetric and positive definite (positive eigenvalues).

Assume that the elastic solid is described, see (1.1–1.5), by $\vec{\mathbf{x}} = \mathbf{G}(t) \vec{\mathbf{s}}$, with inverse $\vec{\mathbf{s}} = \mathbf{H}(t) \vec{\mathbf{x}}$, where $\mathbf{H} = \mathbf{G}^{-1}$ (also symmetric and positive definite).

This corresponds to:

$$\text{velocity field } \dagger (\vec{\mathbf{v}} = d\vec{\mathbf{x}}/dt \text{ with } \vec{\mathbf{s}} \text{ constant}) \quad \vec{\mathbf{v}} = \frac{d\mathbf{G}}{dt} \mathbf{H} \vec{\mathbf{x}} = -\mathbf{G} \frac{d\mathbf{H}}{dt} \vec{\mathbf{x}}, \quad (3.3)$$

$$\text{strain tensor (see (1.11))} \quad \boldsymbol{\sigma} = \frac{1}{2} (\mathbf{G}^2 - \mathbf{1}), \quad (3.4)$$

$$\text{and density} \quad \rho = \rho_0 \det(\mathbf{H}). \quad (3.5)$$

\dagger Here we use that $\frac{d\mathbf{G}}{dt} \mathbf{H} + \mathbf{G} \frac{d\mathbf{H}}{dt} = \mathbf{0}$, which follows from $\mathbf{G} \mathbf{H} = \mathbf{1}$.

The stress tensor depends on the deformations only, (i.e.: \mathbf{G}), thus:

$$\boldsymbol{\tau} \text{ is a function of time only: } \boldsymbol{\tau} = \boldsymbol{\tau}(t).$$

Recall that $\boldsymbol{\tau}$ is symmetric as well.

Remark 3.6 Since \mathbf{G} can be selected to take any positive definite symmetric matrix value, it should be clear [see (1.6) and (1.11)] that $\boldsymbol{\sigma}$ can be made to take any possible value that a strain tensor can have.

These are the **tasks in this problem**:

- A. Show** that the equation for the conservation of mass, is satisfied

$$\rho_t + \text{div}(\rho \vec{\mathbf{v}}) = \mathbf{0}, \quad (3.6)$$

- B. Show** that appropriate body forces $\vec{\mathbf{f}}$ can be selected to make the equation for momentum conservation, apply. **Compute** $\vec{\mathbf{f}}$.

$$(\rho \vec{\mathbf{v}})_t + \text{div}(\rho \vec{\mathbf{v}} \otimes \vec{\mathbf{v}}) = \text{div}(\boldsymbol{\tau}) + \rho \vec{\mathbf{f}}, \quad (3.7)$$

Note. \otimes is the tensor product. It is defined by:

$$\vec{\mathbf{p}} \otimes \vec{\mathbf{q}} = \{p_i q_j\}, \text{ for any vectors } \vec{\mathbf{p}} \text{ and } \vec{\mathbf{q}}.$$

- C. Use the equation for the conservation of energy,**

$$(\rho E)_t + \text{div}(\rho \vec{\mathbf{v}} E) = \text{div}(\boldsymbol{\tau} \cdot \vec{\mathbf{v}}) + \rho \vec{\mathbf{f}} \cdot \vec{\mathbf{v}}, \quad (3.8)$$

to derive a relationship between $\frac{dV}{dt}$ and $\boldsymbol{\tau}$. Here

$V = V(\boldsymbol{\sigma})$ is the elastic deformation energy per unit mass (potential energy density), so that in (3.8) the energy per unit mass is given by:

$$E = \frac{1}{2} \vec{\mathbf{v}} \cdot \vec{\mathbf{v}} + V. \quad (3.9)$$

Note that V is also a function of time only.

Specifically, the relationship that you are expected to justify is

$$\rho \frac{dV}{dt} = \text{Tr} \left(\boldsymbol{\tau} \mathbf{H} \frac{d\boldsymbol{\sigma}}{dt} \mathbf{H} \right). \quad (3.10)$$

Remark 3.7 The body forces needed for (3.7) are (generally) impossible to achieve, globally, for \mathbf{G} arbitrary. However, any smooth configuration can be approximated by this situation in a sufficiently small region. Hence equation (3.10) gives us information about the dynamics that relates the stress tensor to the elastic potential energy. ♣

Hints. The following facts will be useful: Let $\mathbf{U} = \mathbf{U}(t)$ be a differentiable matrix valued function, with \mathbf{U} nonsingular. Then, in general, $\frac{d\mathbf{U}}{dt}$ and \mathbf{U} do not commute. However

1. $\frac{dU^2}{dt} = \frac{dU}{dt} U + U \frac{dU}{dt}$.
2. $\frac{dU^{-1}}{dt} = -U^{-1} \frac{dU}{dt} U^{-1}$.
3. $\dot{y}/y = \frac{d(\ln y)}{dt} = \text{Tr} \left(U^{-1} \frac{dU}{dt} \right) = \text{Tr} \left(\frac{dU}{dt} U^{-1} \right)$, where **Tr is the trace of a matrix** and $y = \det(U)$.

4 Elasticity: Energy considerations

4.1 Potential energy

Elastic deformations have a potential energy associated with them. This energy should be a function of the media deformations only. Hence

$$V = \text{elastic energy per unit mass} = V(\boldsymbol{\sigma}). \quad (4.1)$$

Remark 4.8 A question/ambiguity would seem to arise here: *Which one of V (elastic energy per unit mass), or ρV (elastic energy per unit volume), should be a function of the media deformations (i.e.: σ)?* The answer is **both**, there is no ambiguity. The reason is that the density can be written in terms of the equilibrium density (a constant), and the volume change. The volume change, in turn is encoded in the strain tensor. The key formulas are $\rho = \rho_0 \det(H)$ and $\det(\mathbf{2}\boldsymbol{\sigma} + \mathbf{1}) = (\det(H))^{-2}$. These formulas follow because the ratio of the volumes (equilibrium/current) is $\det(H) > 0$ (equation (1.5)), from the definition of $\boldsymbol{\sigma}$ in (1.11), and the fact that $H = G^{-1}$. Finally, note that for infinitesimal deformations $\det(H) = 1$, so there are no volume changes. ♣

For an **isotropic elastic solid** V should be invariant under rotations — see (1.16), hence **only a function of**

$$\text{Tr}(\boldsymbol{\sigma}) = p_1(\boldsymbol{\sigma}) + p_2(\boldsymbol{\sigma}) + p_3(\boldsymbol{\sigma}), \quad (4.2)$$

$$\text{Tr}(\boldsymbol{\sigma}^2) = p_1^2(\boldsymbol{\sigma}) + p_2^2(\boldsymbol{\sigma}) + p_3^2(\boldsymbol{\sigma}), \quad (4.3)$$

$$\text{Tr}(\boldsymbol{\sigma}^3) = p_1^3(\boldsymbol{\sigma}) + p_2^3(\boldsymbol{\sigma}) + p_3^3(\boldsymbol{\sigma}), \quad (4.4)$$

where the $p_j(\boldsymbol{\sigma})$ are the **principal strains** (eigenvalues of $\boldsymbol{\sigma}$). Note that an alternative to $\text{Tr}(\boldsymbol{\sigma}^3)$ in (4.4) is given by the determinant: $\det(\boldsymbol{\sigma}) = p_1(\boldsymbol{\sigma}) p_2(\boldsymbol{\sigma}) p_3(\boldsymbol{\sigma})$. However, $\text{Tr}(\boldsymbol{\sigma}^3)$ is easier to differentiate, so it has some analytical advantages. In fact, nice functional dependence is the main reason to use (4.2–4.4), instead of the p_j .

In particular: **in the small deformation limit** V should be a quadratic function of $\boldsymbol{\sigma}$, hence

$$\rho V = \frac{1}{2} \lambda (\text{Tr}(\boldsymbol{\sigma}))^2 + \mu \text{Tr}(\boldsymbol{\sigma}^T \boldsymbol{\sigma}), \quad (4.5)$$

where ρ is the mass density, and (λ, μ) are the same constants as in (7.4) — as shown below in (4.8–4.9). Note that, *in this case*, $\rho = \rho_0$ (no volume changes); hence both V and ρV are quadratic functions of $\boldsymbol{\sigma}$ — see remark 4.8.

Remark 4.9 Extension of $V = V(\boldsymbol{\sigma})$ to non-symmetric $\boldsymbol{\sigma}$. The elastic energy V is defined for symmetric tensors only (i.e.: no elastic energy can be attributed to a tensor that does not correspond to a deformation). Yet, it is often convenient to operate as if $\boldsymbol{\sigma}$ was an arbitrary tensor, with its entries un-restricted [for example, this makes it easier when using the chain rule of differentiation]. To do this we

$$\text{“extend” } V \text{ to a function defined for any tensor: } V(\boldsymbol{\sigma}) = V_*(\tfrac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T)),$$

where V_* is the “regular” elastic energy, defined for symmetric tensors only. ♣

This is a mathematical trick; however: once you think of V as a function of $\tilde{\boldsymbol{\sigma}} = \frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^T)$, not only it is obvious that M (defined below) is symmetric (and a function of $\tilde{\boldsymbol{\sigma}}$ as well), but you can freely use the chain rule as if there were no restrictions on $\boldsymbol{\sigma}$.

(1) In index notation: V_* is a function of σ_{ij} for $i \leq j$ only. To get V we replace the argument σ_{ij} by $(\sigma_{ij} + \sigma_{ji})/2$, when $i < j$.

(2) Other ways of doing this may exist. This one has the “advantage” (?) that M is symmetric, even when $\boldsymbol{\sigma}$ is not.

Let now M be **the symmetric tensor** ($M_{ij} = M_{ji}$) **defined by**
Below we show that M mediates the relationship between V and τ .

$$M_{ij} = \frac{\partial}{\partial \sigma_{ij}} V. \quad (4.6)$$

4.2 The relationship between V and τ

Here we will show that

$$\rho M = H \tau H^T. \quad \text{Equivalently: } \tau = \rho G M G^T. \quad (4.7)$$

Hence, **in equation (1.12),** $\tilde{N} = \rho M$.

In the **small deformations limit** (see example 1.1) we approximate: †

$G = \text{identity}$, $\rho = \rho_0$, and $\sigma = \frac{1}{2}(D + D^T)$;

where D is defined in (1.8). Then (4.7) becomes

$$\tau_{ij} = \rho \frac{\partial}{\partial \sigma_{ij}} V. \quad (4.8)$$

Using this for V as in (4.5) yields

$$\tau = \lambda \text{Tr}(\sigma) + 2\mu \sigma, \quad (\text{isotropic case}) \quad (4.9)$$

which is the same as (7.4). Hence the constants in (4.5) and (7.4) are the same.

† We have $G = 1 + D + O(\epsilon^2)$, where $D = O(\epsilon)$ and $0 < \epsilon \ll 1$ — see (1.8). Then, to leading order, $G = 1$, $\rho = \rho_0 / \det(G) = \rho_0$, and $\sigma = \frac{1}{2}(G^T G - 1) = \frac{1}{2}(D + D^T)$.

Note that, generally, $\rho = \rho(\bar{x})$ — since $\rho_0 = \rho_0(\bar{s})$, and $\bar{x} = \bar{s}$ **in this limit.**

Remark 4.10 Why should the stress tensor, and the gradient of the elastic deformation energy density with respect to the strain, be related — as in (4.6–4.7) above? The reason(s) are:

- (i) V has to be the result of the work done by the forces (as described by τ) when deforming the body from equilibrium to some configuration.
- (ii) Work is force (characterized by τ) times distance (characterized by σ).

Note that the total work in item (i) has to be independent of the “deformation path” by which the media is taken from rest to a particular configuration. This is automatically guaranteed by the fact that the forces are given via the gradient of a single function, V . *An alternative way to say this is: The condition that the work done be independent of the deformation-path, so that an elastic energy can be defined, imposes restrictions on the stress tensor. These restrictions are expressed by (4.7).* ♣

Proof of (4.7). Conservation of mass, momentum, and energy yield the equations: †

[Ma] $\rho_t + \text{div}_e(\rho \vec{v}) = 0$, **[Mo]** $(\rho \vec{v})_t + \text{div}_e(\rho \vec{v} \otimes \vec{v}) = \text{div}_e(\tau) + \rho \vec{f}$, and **[En]** $(\rho E)_t + \text{div}_e(\rho \vec{v} E) = \text{div}_e(\vec{v} \cdot \tau) + \rho \vec{f} \cdot \vec{v}$, where (i) $E = \frac{1}{2} \vec{v} \cdot \vec{v} + V$, (ii) $\vec{v} \otimes \vec{v}$ is the tensor with components $(\vec{v} \otimes \vec{v})_{ij} = v_i v_j$, and (iii) $(\text{div}_e(a))_i = (a_{ij})_{x_j}$ for any tensor with components a_{ij} . **Note that:** (a) We use the repeated index summation convention, and (b) We use the subindex e to make it explicit that the derivatives are in the Eulerian frame.

† Verifying that these equations apply is a task left to the reader.

Using **[Ma]** to eliminate ρ_t in **[Mo]** and **[En]** yields: † **[Mo2]** $\rho \frac{D\vec{v}}{Dt} = \text{div}_e(\tau) + \rho \vec{f}$ and **[En2]** $\rho \frac{DE}{Dt} = \text{div}_e(\vec{v} \cdot \tau) + \rho \vec{f} \cdot \vec{v}$, where $\frac{D}{Dt} = \partial_t + \vec{v} \cdot \nabla_e$. **Note that** $\frac{D}{Dt}$ is the same as ∂_t in the Lagrangian frame.

Now substitute the formula for E into **[En2]**, and subtract from the

resulting equation the scalar product of **[Mo2]** with \vec{v} . This yields: † **[En3]** $\rho \frac{DV}{Dt} = \text{Tr}(\tau \cdot q)$, where $q_{ij} = \frac{1}{2} \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$.

Now: #1 $\frac{\partial v_i}{\partial x_j} = \frac{\partial v_i}{\partial s_n} \frac{\partial s_n}{\partial x_j} = ((X_i)_t)_{s_n} H_{nj} = (G_t H)_{ij}$. } Hence: #2 **[R1]** $\text{Tr}(\tau q) = \text{Tr}(\tau G_t H) = \text{Tr}(H \tau G_t)$.

It follows that $q = \frac{1}{2} (G_t H + H^T G_t^T)$.

Furthermore, using the definition of M and σ , and the fact that M is symmetric: #3 **[R2]** $\frac{DV}{Dt} = \text{Tr}(M \frac{D\sigma}{Dt}) = \text{Tr}(M G^T G_t)$.

#1 Here we use the definition of \vec{v} . Note that the time derivatives here are in the Lagrangian frame.

#2 The symmetry of τ collapses the two terms in q into one. Then we use that $\text{Tr}(ab) = \text{Tr}(ba)$.

#3 The time derivative has two terms, transposes of each other. The symmetry of M collapses them into one.

Finally, substitute [R1] and [R2] into [En3], to obtain: [En4] $\text{Tr}((\rho M G^T - H \tau) G_t) = 0$. This has to be satisfied for **all** solutions, which means that it must be: #4 $\rho M G^T - H \tau = 0$. This is (4.7). QED

#4 The argument here is: The full evolution of $\vec{X}(\vec{s}, t)$ is determined by [Mo2] — see remark 4.11, which admits initial values prescribed for $\vec{X}(\vec{s}, 0)$ (initial deformation) and $\vec{X}_t(\vec{s}, 0)$ (initial velocity). Alternatively: because the forces \vec{f} in [Mo2] can be “arbitrary”, \vec{X} and \vec{v} are independent. It follows that G and G_t are independent. Hence, an equation of the form $\text{Tr}(\text{function}(G) G_t) = 0$ can apply for all solutions if and only if $\text{function}(G) = 0$.

Remark 4.11 What exactly have we proved above? The “fundamental” dependent variable in this problem is $\vec{X}(\vec{s}, t)$, since everything else can be written in terms of it. Given a stress tensor, the time evolution of \vec{X} is completely determined by the conservation of momentum, i.e.: [Mo], or [Mo2] — the second because mass is conserved “by construction” (i.e.: $\rho = \rho_0 \det(H)$), so one can use [Ma] to simplify [Mo]. Then the conservation of angular momentum is equivalent to τ being symmetric — this is shown in §2. Finally, what we have shown here is that conservation of energy is equivalent to (4.7). ♣

4.2.1 The general isotropic media case

For an isotropic media, see (4.2–4.4), we have $V = V(\zeta_1, \zeta_2, \zeta_3)$, where $\zeta_n = \frac{1}{n} \text{Tr}(\sigma^n)$. (4.10)

Then (4.6–4.7) yields

$$\tau = \rho G (\alpha_1 + \alpha_2 \sigma + \alpha_3 \sigma^2) G^T, \quad \text{where } \alpha_n = V_{\zeta_n}. \quad (4.11)$$

In the small deformations case: $\alpha_1 = \lambda \text{Tr}(\sigma)$, $\alpha_2 = 2\mu$, and $\alpha_3 = 0$ — see (4.5), (4.8) and (4.9).

5 Elasticity: The 1-D case

Here we consider the *longitudinal vibrations of an elastic rod*, for which we make the approximation that motion/deformation occurs only † along the axis of the rod.

† Note that this is an idealization, because when the stress is unidirectional, some strain has to occur in the transversal directions, as shown in (7.10–7.11).

Introduce a cartesian coordinate system, x , along the rod’s axis. Then we can describe the state of the rod via the function $x = X(s, t)$, with inverse $s = S(x, t)$. Here x is the position of a rod “slice” perpendicular to the axis, and s is a label for the slices, defined by: $x = s$ when the rod is at equilibrium [no forces]. Note that s is then a Lagrangian coordinate, attached to the mass elements (slices) along the rod.

Following the notation from prior sections, introduce

1. The functions: $G = \partial X / \partial s$ and $H = \partial S / \partial x$ — see (1.5). Units: none.
Note that: $G = 1/H > 0$ [rod cannot be deformed and “pushed” through itself].
2. The strain tensor: $\sigma = \frac{1}{2} (G^2 - 1)$ — see (1.11). Note that $\sigma > -1/2$. Units: none.
3. The density: $\rho = \rho_0(s) H$, where ρ_0 is the equilibrium density — see (3.5). Units: mass/length.
4. The elastic energy per unit mass: $V = V(\sigma)$ — see (4.1). Units: velocity².
5. The stress tensor: $\tau = \rho G^2 dV/d\sigma$. — see (4.7). Units: force.
Of course, in this case *the stress tensor is just the **tension** along the rod.*

In this 1-D case it is *convenient to think of τ and V as functions of $G = \sqrt{1 + 2\sigma}$, instead of σ* . (5.1)

Remark 5.12 *Elastic forces in 1-D are most easily/intuitively understood in terms of how much force a relative change in length generates* (i.e.: each small section of the rod behaves like a spring). That is: consider a small section of the rod of length L at equilibrium, stretched/contracted to a length $L + \delta L$. Then the force should be a function

of $\delta L/L$ — in fact: proportional to $\delta L/L$ when Hooke's law applies. However, *note that* $\delta L/L = G - 1$ in the limit $L \rightarrow 0$. ♣

Remark 5.13 We use this example to make a direct/intuitive verification of the formula for τ in item 5 above. The *relationship between V and τ* can be obtained by computing how much work does it take to stretch (or compress) a small rod segment of equilibrium length L to some other length L_1 . Let the change be done by keeping the left end fixed and moving right end at constant speed v_r . If the segment is small enough we can assume that the length change is uniform (thus τ and V are constants along the segment at any time, and $\rho_0 = \text{constant}$). Then the rod segment is described by

$$x = X(s, t) = (1 + (v_r/L)t) s, \quad (5.2)$$

where $0 \leq s \leq L$, and the right end of the segment is at $x_r = L + v_r t$.

The total mass in the segment is $\rho_0 L$, so that the elastic energy in it is $\rho_0 L V(G)$, where $G = 1 + (v_r/L)t$. On the other hand, work is being done † at the rate $\tau(G) v_r$. Since

$$\rho_0 L V(G_1) = \int_0^{t_1} \tau(G(t)) v_r dt, \quad (5.3)$$

the elastic energy vanishes at equilibrium, we have where t_1 is the time at which $x_r = L_1$, and $G_1 = G(t_1)$.

† For example, by a force applied to the right end of the segment, to change its length.

Changing variables, using $v_r dt = L dG$, yields: $\rho_0 V(G_1) = \int_1^{G_1} \tau(G) dG$. Hence $\tau = \rho_0 \frac{dV}{dG}$. (5.4)

Using $\sigma = \frac{1}{2}(G^2 - 1)$ and $\rho G = \rho_0$ item 5 above follows. **QED**

Note that it should be $\tau(\mathbf{1}) = \mathbf{0}$, hence $G = \mathbf{1}$ *should be an extremum of V* , in fact: a **minimum**, see (5.6). ♣

5.1 Equations of motion

A standard conservation law argument shows that conservation of momentum leads to the equation

$$\rho_0 \mathbf{X}_{tt} = \boldsymbol{\tau}_s = \gamma(\mathbf{X}_s) \mathbf{X}_{ss}, \quad \text{where } \gamma = \frac{d\boldsymbol{\tau}}{d\mathbf{G}} \quad (\text{note } \mathbf{G} = \mathbf{X}_s). \quad (5.5)$$

Note that this equation is in Lagrangian coordinates — i.e.: in terms of the reference frame for the media.

Remark 5.14 This equation “makes sense” only for $\gamma \geq 0$, *in which case it is a nonlinear wave equation, with speed $c^2 = \gamma/\rho_0$* . If this is violated the equation becomes ill-posed as an evolution in time, which means that the model fails: at least one of the physical assumptions leading to (5.5) no longer applies, and another model is needed.

Note that $\gamma < 0$ means that (for $G > 0$) the resistance of the rod to stretching goes down as the rod is stretched, and (for $G > 0$) the resistance to compression goes down as the rod is compressed. These are very unstable situations. They actually happen when the elasticity limit is reached (too much stretching or compression) and, for example, plastic deformation starts. Then standard elasticity no longer applies. ♣

For a “normal” elastic media $\gamma = \rho_0 c^2 > 0$. This, coupled with $\tau(\mathbf{1}) = \mathbf{0}$ and (5.5), means that

$$V \text{ is a strictly concave function of } G, \text{ with minimum at } G = 1. \quad (5.6)$$

5.1.1 Other conservation laws

Conservation of mass is automatic, by construction (in the Lagrangian reference frame, the density ρ_0 is time independent). As for the conservation of energy, it is easy to check that (5.4) and (5.5) lead to

$$(\rho_0 \mathbf{E})_t - (\mathbf{X}_t \boldsymbol{\tau})_s = \mathbf{0}, \quad \text{where } \mathbf{E} = \frac{1}{2} \mathbf{X}_t^2 + V. \quad (5.7)$$

An alternative way to derive (5.4) is: (5.5) and (5.7) must both apply, which implies (5.4).

5.1.2 Eulerian coordinates

We can also write the equations with (x, t) as the independent variables. For this purpose we introduce $G = X_s$, $v = X_t$, and $\rho = \rho_0/G$, as the dependent variables. Next we note the following three equations (valid in the Lagrangian coordinates):

$$(\rho G)_t = 0, \quad G_t - v_s = 0, \quad \text{and} \quad G \rho v_t - \tau_s = 0.$$

Then we use the derivative transformation rules: $\partial_t \rightarrow \partial_t + v \partial_x$ and $\partial_s \rightarrow G \partial_x$ to get the equations:

$$\rho_t + (\rho v)_x = 0, \tag{5.8}$$

$$v_t + v v_x - \frac{1}{\rho} \tau_x = 0, \tag{5.9}$$

$$G_t + v G_x - G v_x = 0, \tag{5.10}$$

with $\tau = \tau(G)$. Note that, when $\rho_0 = \text{constant}$, $G = \rho_0/\rho$ and we do not need the third equation. Then the other two equations are the same as the equations of isentropic Gas Dynamics, with $p = -\tau$ the pressure. In general, the third equation can also be written in terms of $H = 1/G$; then:

$$H_t + (v H)_x = 0.$$

Combining (5.8) and (5.9) yields the equation for the

conservation of momentum in Eulerian coordinates:

$$(\rho v)_t + (\rho v^2 - \tau)_x = 0.$$

6 Problem: Conservation laws in elasticity

6.1 Statement: Conservation laws in elasticity

Note: despite the problem title, this exercise applies to any continuum media.

Consider a continuum media (solid or liquid) in some region of space, with an inertial cartesian coordinate system \vec{x} . Let the media have the following fields associated with it: $\rho = \rho(\vec{x}, t)$ = mass density, $\vec{v} = \vec{v}(\vec{x}, t)$ = mass flow velocity, $\tau = \tau(\vec{x}, t)$ = stress tensor, $\vec{f} = \vec{f}(\vec{x}, t)$ = body forces per unit mass, and $V = V(\vec{x}, t)$ = internal energy per unit volume. Assume that all these fields are smooth functions of (\vec{x}, t) , and:

1. **Derive pde that the fields above should satisfy.** Use conservation of mass, (linear) momentum, and energy.
2. Use the equations in item 1 to **obtain expressions for the flow velocity and internal energy material derivatives.** That is, for:

$$\frac{D\vec{v}}{Dt} = \vec{v}_t + (\vec{v} \cdot \nabla)\vec{v} \quad \text{and} \quad \frac{DV}{Dt} = V_t + (\vec{v} \cdot \nabla)V.$$

Note that in the equation for $\frac{DV}{Dt}$ the body forces should not appear.

3. **Write the equation for the conservation of angular momentum** and show that, given the equations in item 1, it is **satisfied if and only if τ is symmetric.**

Remark 6.15 The equations above have to be completed with appropriate equations of state. For example, in gas dynamics τ is given in terms of the pressure p and \vec{v} , while V is given in terms of p and ρ . In elasticity τ is a function of the strain tensor. ♣

Recall that the stress tensor is defined as follows: for any surface with unit normal \hat{n} , the force per unit area (by the media on the side the normal points towards, onto the other side) is given by $\tau \cdot \hat{n} = \{\tau_{ij} n_j\}$ — where **we use the repeated index summation convention.**

Important: for parts 1-2 be careful to **not** use the symmetry of τ , since then part 3 would become pointless.

Notation: Let \vec{a} and \vec{b} be vectors, \mathbf{c} a rank-two tensor, and ε_{knm} the permutation multi-index. † Then

- (1) $\text{div}(\vec{a}) = (a_j)_{x_j}$ — a scalar.
- (2) $\text{div}(\mathbf{c}) = \{(c_{ij})_{x_j}\}$ — a vector.
- (3) $\vec{a} \otimes \vec{b}$ is a rank-two tensor defined by $(\vec{a} \otimes \vec{b})_{ij} = a_i b_j$.
- (4) $\vec{a} \cdot \mathbf{c} = \{a_i c_{ij}\}$ and $\mathbf{c} \cdot \vec{a} = \{c_{ij} a_j\}$ — both vectors. Generally, $\vec{a} \cdot \mathbf{c} \neq \mathbf{c} \cdot \vec{a}$ (unless \mathbf{c} is symmetric).
- (5) $\vec{a} \times \vec{b} = \{\varepsilon_{jnm} a_n b_m\}$ — a vector.
- (6) $\vec{a} \times \mathbf{c}$ and $\mathbf{c} \times \vec{a}$ are rank-two tensors defined by $(\vec{a} \times \mathbf{c})_{ij} = \{\varepsilon_{inm} a_n c_{mj}\}$, and $(\mathbf{c} \times \vec{a})_{ij} = \{\varepsilon_{inm} c_{jn} a_m\}$.
 $\dagger \varepsilon_{knm}$ is defined by: $\varepsilon_{knm} = 1$ if $(k, n, m) = (1, 2, 3), (2, 3, 1), (3, 1, 2)$; $\varepsilon_{knm} = 0$ if two of the indexes are equal; $\varepsilon_{knm} = -1$ otherwise.

7 Elasticity: The small deformations limit

Here we **assume that D in (1.8) is small** — $O(\epsilon)$ with $0 < \epsilon \ll 1$, and neglect $O(\epsilon^2)$ terms.

Then, since $\vec{s} = \vec{x} - \vec{u}$, $H = 1 - D$ with $D_{ij} = (u_i)_{x_j}$.[†]

But $G = H^{-1} = 1 + D + O(\epsilon^2)$. Thus, **in this limit**

$$H = 1 - D \quad \text{and} \quad G = 1 + D. \quad (7.1)$$

[†] Recall: all of these variables are introduced in (1.1–1.11).

Furthermore

$$\vec{s} = \vec{x} \quad \text{and} \quad \sigma = \frac{1}{2} (D + D^T), \quad (7.2)$$

as follows from $2\sigma = G^T G - 1$.

Remark 7.16 In this limit the assumption is that the displacements are small (in fact: infinitesimal), so that the distinction between \vec{s} and \vec{x} disappears. However, *it is possible to have small deformations, while having significant displacements* — e.g.: A thin flat plate (or a thin rod) can undergo significant lateral displacements while keeping the deformations small.

This situation occurs when dG_s in (1.6) is close to the identity. That is, when $G = R(1 + D)$, where R is a rotation and $D = D^T$ is small/infinitesimal. In this case the distinction between \vec{s} and \vec{x} remains, but Hooke's law can still be used for the strain-stress relationship. ♣

Remark 7.17 Relationship between V and τ , and isotropy.

Because in this limit $G = H = 1$ at leading order, (4.7) becomes

$$\tau_{ij} = \rho \frac{\partial}{\partial \sigma_{ij}} V, \quad (4.8)$$

where $\rho = \rho_0(\vec{x})$. Similarly, the **conditions**

for an isotropic media in §1.2.3 reduce to[†]

$$\tau(R_l \sigma R_l^T) = R_l \tau(\sigma) R_l^T, \quad (7.3)$$

where R_l is a rotation in the reference space. Of course, must also hold — note that this last identity, together with (4.8) yields (7.3).

[†] To prove (7.3), note that we cannot directly apply here the arguments in §1.2.3, because a rotation in reference space would destroy the small deformation limit in which $G \approx 1$. What we need to do is simultaneously apply a counter-rotation in physical coordinates, so that $G \rightarrow R_l G R_l^T$ — this causes both: $D \rightarrow R_l D R_l^T$ and $\sigma \rightarrow R_l \sigma R_l^T$. Then τ does not change because of the rotation in configuration space, but it changes as in (7.3) because of the counter-rotation — see item **A** in §1.2.1.

An consequence of the small deformation limit here is that **τ is a function of σ only: $\tau = \tau(\sigma)$** . This is not the case in the general case, where a separate dependence on G is also needed — see (1.12). ♣

7.1 Hooke's law

If σ is small, and τ is a smooth function of σ , we can **approximate τ by a linear function of σ : Hooke's law**. This requires 36 coefficients (because the symmetric tensors τ and σ involve 6 independent entries each). However, **if the**

media is isotropic, the relationship between σ and τ has to be invariant under rotations — that is: (7.3) applies.

This reduces the number of free

parameters to two (see §7.1.2), so that †

$$\tau = \lambda \text{Tr}(\sigma) + 2\mu \sigma, \quad (7.4)$$

where $\lambda > 0$ and $\mu > 0$ are the **Lamé constants** (μ is

the **shear modulus**), σ is as in (7.2), and (1.8) yields

$$\text{Tr}(\sigma) = \text{div}(\vec{u}) = \frac{\partial u_i}{\partial x_i}. \quad (7.5)$$

† For more on this formula, see remark 7.18 and §4.

Notes:

A. See remark 7.19 for the inequalities $\lambda, \mu > 0$. In particular, regarding $\lambda > 0$, note that $\lambda > -(2/3)\mu$ is the only “hard” restriction, though for most common materials $\lambda > 0$.

B. For a non-homogeneous solid, $\lambda = \lambda(\vec{x})$ and $\mu = \mu(\vec{x})$.

Remark 7.18 Equation (7.4) implies **stress and strain colinearity**: τ and σ **have the same principal axes**. At every point there is a rotation matrix R such that both $R\tau R^T$ and $R\sigma R^T$ are diagonal.

Conversely, **if colinearity is assumed, (7.4) follows**. In this case the problem reduces to characterizing all the linear transformations between 3 by 3 diagonal matrices, with the property that the transformations are invariant under permutations of the diagonal elements. The most general linear transformation between 3 by 3 diagonal matrices involves 9 coefficients c_{ij} — defined by the equations $d(\tau)_i = c_{ij} d(\sigma)_j$, where $d(\tau)_i$ and $d(\sigma)_j$ are the diagonal elements. Further: there are 6 permutations to enforce invariance under. A little work then shows that it must be $c_{ij} = a$ if $i = j$ and $c_{ij} = b$ if $i \neq j$, for some constants a and b . This is (7.4), with $a = \lambda + 2\mu$ and $b = 2\mu$.

That colinearity should apply is intuitively “obvious”. Consider a deformation that involves no rotation (pure shear, definition 1.4) — there is no loss of generality in this, because the stress tensor ignores rotations. Observe now what happens with an infinitesimal cube whose facets are normal to the principal axes of strain: the cube is deformed in such a way that each facet moves normal to itself by some amount determined by the corresponding principal strain. Under these conditions, how can shear stresses (parallel to the facets) arise? Such a situation would not be invariant under rotation, for the shear stresses would determine special directions in space which are not encoded within σ . Hence no such shear stresses can arise, meaning that colinearity applies. Unfortunately, this argument is not 100% tight. † Another argument leading to (7.4), using the relationship between the stress tensor and the elastic energy stored in the solid, is presented in: §4, equation (4.9). ♣

† The argument only works in the small deformation limit, because of the merging of \vec{s} and \vec{x} into one single system, and the consequences that this has (e.g.: see remark 7.17). In fact, **colinearity fails in the general case** — see (4.11).

Remark 7.19 Should the Lamé constants in (7.4) be positive? This cannot follow from symmetry arguments. It (partly) follows from the fact that the stresses generated by an elastic solid oppose the deformations that cause them: in order to deform an elastic solid, energy must be spent.³ This spent energy then goes into stored elastic deformation energy. In §4, equation (4.5), we show that **the elastic deformation energy V** (per unit mass) associated with (7.4) **has the form**

$$\rho V = \frac{1}{2} \lambda (\text{Tr}(\sigma))^2 + \mu \text{Tr}(\sigma^T \sigma) = \frac{1}{2} \lambda (\sum_i p_i(\sigma))^2 + \mu \sum_i p_i^2(\sigma), \quad (7.6)$$

where the $p_i(\sigma)$ are the principal strains (eigenvalues of σ), and ρ is the density. From the arguments above, **it should be $V > 0$ for any non-trivial σ** . In other words, **the quadratic form in (7.6) should be positive definite**.

A simple calculation (see below) shows

that this is equivalent to

$$\mu > 0 \quad \text{and} \quad 3\lambda + 2\mu > 0, \quad (7.7)$$

which **allows λ non-positive** in the range $-(2/3)\mu < \lambda \leq 0$.

See the *Poisson ratio discussion* in §7.1.1. ♣

³ The opposite would give rise to very unstable (and rather un-physical) situations, with deformations growing without bound in the absence any applied forcing. Within the linear regime, energy out of nothing!

Calculation leading to (7.7). We show that:

$$Q = a(x + y + z)^2 + b(x^2 + y^2 + z^2) \text{ is positive definite} \iff 3a + b > 0 \text{ and } b > 0. \quad [A]$$

Proof of \Rightarrow . (1) Take $y = z = -x/2 \neq 0$. Then $Q > 0$ shows that $b > 0$.

(2) Take $x = y = z \neq 0$. Then $Q > 0$ shows that $3a + b > 0$.

Proof of \Leftarrow . If $a \geq 0$ and $b > 0$, it is obvious that $Q > 0$. Thus assume $b > 0$ and $-b/3 < a < 0$. Then

$Q \geq (3a + b)(x^2 + y^2 + z^2)$, using (#) below. Thus $Q > 0$ as well.

$$\text{(\#)} \quad 3(x^2 + y^2 + z^2) - (x + y + z)^2 = (x - y)^2 + (y - z)^2 + (z - x)^2 \geq 0.$$

7.1.1 The other constants

The **bulk modulus** $B = \lambda + \frac{2}{3}\mu > 0$ relates the traces of τ and σ .

That is, from (7.4)

$$\text{Tr}(\tau) = 3B \text{Tr}(\sigma). \quad (7.8)$$

Note that:

A. The **pressure** (the shear-less component of the stress) is given by

$$p = -\frac{1}{3} \text{Tr}(\tau).$$

B. The **volume change relative to the equilibrium state** is given by as follows from the definition of G in (1.5).

$$\Delta \text{Vol} = \det(G) - 1,$$

Now we use (7.1), $G = 1 + D$, the fact that that D is small, and (7.2), to write $\det G = 1 + \text{Tr}(D) = 1 + \text{Tr}(\sigma)$. Hence

$$\Delta \text{Vol} = \text{Tr}(\sigma).$$

Recall that D is defined in (1.8), and that $\text{Tr}(\sigma) = \text{div}(\vec{u})$.

We conclude that

$$p = -B \Delta \text{Vol}. \quad (7.9)$$

By contrast, the **shear modulus** $\mu > 0$ relates the trace-less part of σ (deformation without volume change; i.e.: shear) with the trace-less part of τ (what remains of τ after the pressure is removed).

Now *consider a situation where the stress is uni-directional*, † that is: only one principal stress is non-zero. Then the ratio of the non-zero stress to its corresponding strain is the **Young's modulus** E .

† Example: pull along its axis a thin rod of the elastic material

To get an expression for E , align the coordinate axes with the stress (and strain) tensors principal axes. This leads, using (7.4), to

$$\begin{aligned} E \sigma_{11} &= \tau_{11} = (\lambda + 2\mu) \sigma_{11} + \lambda(\sigma_{22} + \sigma_{33}), \\ 0 &= \tau_{22} = (\lambda + 2\mu) \sigma_{22} + \lambda(\sigma_{11} + \sigma_{33}), \\ 0 &= \tau_{33} = (\lambda + 2\mu) \sigma_{33} + \lambda(\sigma_{11} + \sigma_{22}), \end{aligned} \quad (7.10)$$

where τ_{11} is the non-vanishing principal stress. Thus ‡

$$E = \frac{3\lambda + 2\mu}{\lambda + \mu} \mu > 0 \quad \text{and} \quad \sigma_{22} = \sigma_{33} = -\frac{\lambda}{2(\lambda + \mu)} \sigma_{11}. \quad (7.11)$$

‡ The last two equations in (7.10) are a non-singular system (determinant $4\mu(\lambda + \mu) > 0$) for $(\sigma_{22}, \sigma_{33})$ in terms of σ_{11} .

The **Poisson's ratio** ν is the (negative of) the ratio between the side strains and the main strain in the uni-axial stress situation defining E . From (7.11)

$$-1 < \nu = \frac{\lambda}{2(\lambda + \mu)} < \frac{1}{2}, \quad (7.12)$$

where the inequalities follow from (7.7). *The Poisson ratio is a measure of the transverse contraction (or expansion) when the material is under uni-axial stress: $\nu > 0$ corresponds to contraction when the material is under tension* (stretched), both τ_{11} and σ_{11} positive, and expansion in the opposite situation. **Most material behave like this**, hence (in general)

$$\nu > 0 \implies \lambda > 0. \quad (7.13)$$

However this is a consequence of the “typical” microscopic structure of common materials, and not of some deep physical principle. In fact **man made materials for which $\nu < 0$, hence $\lambda < 0$, exist.** † These materials have a

rather counter-intuitive behavior: *when a rod made of them is stretched along the axis, its cross-sectional area grows. By the same token, the cross-section decreases when pushing the rod inwards from its ends.*

† They are known by various names, such as: anti-rubber, auxetic, dilatational, or metamaterials.

Finally, notice that we can write

$$\nu = \frac{3\gamma - 2}{2(3\gamma + 1)}, \quad \text{where } \gamma = \frac{B}{\mu}. \quad (7.14)$$

Hence, materials with $\nu < 0$ have γ “small”, and resist shear better than compression. On the other hand, materials that resist compression much better than shearing (e.g.: rubber) have γ “large”, hence ν close to 0.5. An example of a material with ν small (i.e.: γ close to 2/3) is cork — an advantage when making bottle corks, though this is not the main reason for its use.

7.1.2 The consequences of symmetry

Here we show how (7.4) follows from (7.3), and the fact that the transformation $\sigma \rightarrow \tau$ is linear. We begin by considering what τ should be when σ is diagonal.

Using the linearity, we can write

$$\tau(\Lambda) = \zeta_1 A_1 + \zeta_2 A_2 + \zeta_3 A_3 \quad (7.15)$$

when $\Lambda = \text{diag}(\zeta_j)$, where the A_j are some (constant) symmetric tensors.

Now introduce the rotations R_j , which exchange diagonal elements of a diagonal tensor. For example: R_1 exchanges ζ_2 and ζ_3 , and is given by

$$R_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}. \quad \text{Note that: } R_1^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \quad (7.16)$$

Using (7.3) we can then write $\tau(R_\ell \Lambda R_\ell^T) = R_\ell \tau(\Lambda) R_\ell^T = \sum \zeta_j R_\ell A_j R_\ell^T$. On the other hand, since $R_\ell \Lambda R_\ell^T$ is diagonal, we can also use (7.15) directly. Comparing the two expressions, and using the fact that the ζ_j are arbitrary, we obtain

$$R_1 A_1 R_1^T = A_1, \quad R_1 A_2 R_1^T = A_3 \quad \text{and} \quad R_1 A_3 R_1^T = A_2 \quad \left(\implies R_1^2 A_2 (R_1^2)^T = A_2 \quad \text{and} \quad R_1^2 A_3 (R_1^2)^T = A_3 \right)$$

when $\ell = 1$, and similar for the other two cases. Then, from $R_2^2 A_1 (R_2^2)^T = A_1$ and $R_3^2 A_1 (R_3^2)^T = A_1$, we conclude that A_1 is diagonal — similarly A_2 and A_3 must be diagonal. Next we use that $R_1 A_1 R_1^T = A_1$ to conclude that $A_1 = \text{diag}(\mathbf{a}, \mathbf{b}, \mathbf{b})$ for some constants \mathbf{a} and \mathbf{b} . Then $A_2 = R_3 A_1 R_3^T$ and $A_3 = R_2 A_1 R_2^T$ yield $A_2 = \text{diag}(\mathbf{b}, \mathbf{a}, \mathbf{b})$ and $A_3 = \text{diag}(\mathbf{b}, \mathbf{b}, \mathbf{a})$. Now let $\lambda = \mathbf{b}$ and $\mu = (\mathbf{a} - \mathbf{b})/2$, so that (7.15) takes the form

$$\tau(\Lambda) = (\zeta_1 + \zeta_2 + \zeta_3) \lambda + 2\mu \Lambda. \quad (7.17)$$

This is exactly (7.4) for the case $\sigma = \Lambda$. The

general case follows by writing $\sigma = R \Lambda R^T$ and then using (7.3) and (7.17).

The argument above shows that any τ satisfying (7.3) must have the form in (7.4).

Vice-versa, it should be obvious that τ as given by (7.4) satisfies (7.3).

♣

8 Problem: Energy in linear elasticity

The objective of this problem is to verify that the equations of linear elasticity guarantee the conservation of energy.

8.1 Statement: Energy in linear elasticity

In linear elasticity we label each point in the elastic solid by its position at equilibrium, \vec{x} , and describe the dynamics by the **displacement vector** $\vec{u} = \vec{u}(\vec{x}, t)$, which is assumed “infinitesimal”. The *displacement vector is defined as*

follows: the position at time t of the point with label \vec{x} is $\vec{x} + \vec{u}$. Then, for an isentropic⁴ solid the equations are

$$\rho \vec{v}_t = \rho \vec{f} + \operatorname{div}(\boldsymbol{\tau}), \quad (8.1)$$

where $\rho = \rho(\vec{x})$ is the density (constant if the media is homogeneous), $\vec{v} = \vec{u}_t$ is the velocity, $\vec{f} = \vec{f}(\vec{x}, t)$ is the body force per unit mass, $\boldsymbol{\tau}$ is the stress tensor (symmetric), and $\operatorname{div}(\boldsymbol{\tau}) = \{(\tau_{ij})_{x_j}\}$.

Note: we use the repeated index summation convention.

The stress tensor is given by
$$\boldsymbol{\tau} = \lambda \operatorname{div}(\vec{u}) + 2 \mu \boldsymbol{\sigma}, \quad (8.2)$$

where

$$\sigma_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad (8.3)$$

is the strain tensor, and (λ, μ) are the Lamé “constants” — though they may be functions of \vec{x} if the media is not homogeneous. Note that it must be $\mu > 0$ and $\lambda > -(2/3)\mu$.

Equation (8.1) expresses the conservation of momentum, where $\rho \vec{v}$ is the momentum density and the stress tensor (forces) is the negative of the momentum flux. Recall that the stress tensor is defined by: For any unit vector \hat{n} , $\boldsymbol{\tau} \cdot \hat{n}$ is the force per unit area, across a surface normal to \hat{n} , by the solid on the side of the surface \hat{n} points towards, onto the other side.

Write the appropriate equation for the conservation of energy, and verify that it is satisfied.

Hint. The required equation has the form $\mathcal{E}_t + \operatorname{div}(\mathcal{F}) = W$, where \mathcal{E} is the (total) energy density (per unit volume), \mathcal{F} is the energy flux vector, and W is the work per unit volume done by the body force. You will need (8.4) in order to write \mathcal{E} .

The elastic deformation energy per unit mass V is given by
$$\rho V = \frac{1}{2} \lambda (\operatorname{Tr}(\boldsymbol{\sigma}))^2 + \mu \operatorname{Tr}(\boldsymbol{\sigma}^T \boldsymbol{\sigma}). \quad (8.4)$$

9 Euler-Lagrange equations (variational formulation)

In the variational formulation of the equations, we start with the Lagrangian
$$L = \frac{1}{2} \rho_0 \vec{v}^2 - \rho_0 V, \quad (9.1)$$
 formulated in the reference (or Lagrangian) frame, where V is thought as a function of G ,[†] and $\rho_0 = \rho_0(\vec{s}) =$ mass density in the reference frame. Then the Euler-Lagrange equations follow

$$\rho_0 \vec{v}_t = \operatorname{div}_l(\mathcal{T}^{(1)}), \quad \text{where } \mathcal{T}_{ij}^{(1)} = \rho_0 \frac{\partial V}{\partial G_{ij}} = \text{first Piola-Kirchhoff stress tensor. } \ddagger \quad (9.2)$$

[†] Since $\boldsymbol{\sigma} = \frac{1}{2}(\mathbf{G}^T \mathbf{G} - \mathbf{1})$, this does not contradict the fact that V is a function of $\boldsymbol{\sigma}$.

[‡] The second Piola-Kirchhoff stress tensor is introduced in (9.10).

These equations are formulated in the reference frame: div_l stands for the divergence in the \vec{s} variables, and the time derivatives are for \vec{s} constant. **Notation:** we use a subscript l to denote operators in the Lagrangian frame (e.g.: $\nabla_l = (\partial_{s_1}, \partial_{s_2}, \partial_{s_3})^T$) and a subscript e to denote operators in the Eulerian frame. **Recall that:** (i) ∂_t in the Lagrangian frame becomes $\partial_t + \vec{v} \cdot \nabla_e = \frac{D}{Dt}$ in the Eulerian frame. (ii) div applies to the last index in a tensor; e.g.: $\operatorname{div}_l(\mathcal{T}^{(1)}) = \partial_{s_j}(\mathcal{T}^{(1)})_{ij}$ — see note 1.2a in definition 1.2, and the “notation” paragraph below remark 6.15.

Next we use (9.11) to write $\mathcal{T}^{(1)}$ in terms of M , as defined in (4.6).

$$\mathcal{T}_{ij}^{(1)} = \rho_0 M_{pq} \frac{\partial \sigma_{pq}}{\partial G_{ij}} = \frac{1}{2} \rho_0 (M_{jq} G_{iq} + M_{pj} G_{ip}), \quad \text{i.e.: } \mathcal{T}^{(1)} = \rho_0 \mathbf{G} \mathbf{M}, \quad (9.3)$$

where we use the repeated index summation convention and the symmetry of M . Using (4.7) we can write

$$\mathcal{T}^{(1)} = \mathbf{J} \boldsymbol{\tau} \mathbf{H}^T, \quad \text{with } \mathbf{J} = \det(\mathbf{G}) \text{ and } \rho_0 = \mathbf{J} \rho, \quad (9.4)$$

where $\rho =$ mass density in the Eulerian frame — i.e.: the “physical” frame. Note that $\mathcal{T}^{(1)}$ is not symmetric.

⁴ Properties are invariant under rotation.

9.1 Conservation of angular momentum

The angular momentum per unit mass is $\vec{A} = \vec{v} \times \vec{X}$. Thus the vector product of (9.2) with \vec{X} yields

$$(\rho_0 \vec{A})_t = -\vec{X} \times \text{div}_l(\mathcal{T}^{(1)}) = -\text{div}_l(\vec{X} \times \mathcal{T}^{(1)}) + \left\{ \varepsilon_{ipq} G_{pj} (\mathcal{T}^{(1)})_{qj} \right\} = -\text{div}_l(\vec{X} \times \mathcal{T}^{(1)}). \quad (9.5)$$

Here we use: **(i)** ρ_0 does not depend on t ; **(ii)** $\vec{A}_t = \vec{v}_t \times \vec{X}$, because $\vec{X}_t = \vec{v}$; **(iii)** the repeated index summation convention; and **(iv)** $\{G_{pj} (\mathcal{T}^{(1)})_{qj}\} = \mathcal{T}^{(1)} G^T$ is symmetric, as follows from (9.3) — this justifies the last identity.

9.2 Transformation to Eulerian coordinates

Consider an arbitrary (fixed) region, Ω_l , in the reference frame, with boundary $\partial\Omega_l$. Let Ω_e and $\partial\Omega_e$ be the corresponding region, and boundary, in the Eulerian frame (note: Ω_e may vary in time, even though Ω_l does not). Next examine the total momentum balance in the Ω 's. We have:

- #1. $\int_{\Omega_e} \rho \vec{v} \, dV_e = \int_{\Omega_l} \rho_0 \vec{v} \, dV_l$.
- #2. $\frac{d}{dt} \int_{\Omega_e} \rho \vec{v} \, dV_e = \int_{\Omega_e} (\rho \vec{v})_t \, dV_e + \int_{\partial\Omega_e} \rho \vec{v} (\vec{v} \cdot \hat{n}_e) \, dA_e = \int_{\Omega_e} ((\rho \vec{v})_t + \text{div}_e(\rho \vec{v} \otimes \vec{v})) \, dV_e$. †
- #3. $\frac{d}{dt} \int_{\Omega_l} \rho_0 \vec{v} \, dV_l = \int_{\Omega_l} \text{div}_l(\mathcal{T}^{(1)}) \, dV_l = \int_{\partial\Omega_l} \mathcal{T}^{(1)} \cdot \hat{n}_l \, dA_l = \int_{\partial\Omega_e} (\mathcal{T}^{(1)} \cdot G^T \hat{n}_e) J^{-1} \, dA_e$
 $= \int_{\Omega_e} \text{div}_e \left(\frac{1}{J} \mathcal{T}^{(1)} \cdot G^T \right) \, dV_e$.

† **Important:** The partial derivatives with respect to time of the integrands are in the Eulerian frame; thus with \vec{x} frozen.
 † Further: the tensor product \otimes is defined in the "notation" paragraph below remark 6.15.

Proofs (below, in items with more than one equality, (i), (ii), etc. is the argument justifying successive equalities).

- #1. This follows because $\rho_0 = J \rho$, and the volume elements are related by $dV_e = J \, dV_l$.
- #2. (i) Here the second integral follows from the fact that Ω_e changes in time. This integral accounts for the change in momentum as the boundary $\partial\Omega_e$ moves normal to itself at velocity $\vec{v} \cdot \hat{n}_e$. (ii) Use Gauss' theorem.
- #3. (i) Use (9.2) and the fact that Ω_l is fixed. (ii) Use Gauss' theorem. (iii) Use (9.12) to change the integration from $\partial\Omega_l$ to $\partial\Omega_e$. (iv) Use Gauss' theorem. ‡

‡ **Key point here:** Note that if we attempted to jump directly from the second to the last formula in #3, the result would involve div_l , **not** div_e . Since we need div_e , we are forced to introduce two intermediate steps.

From #1 above, it follows that #2 and #3 are equal. But Ω_e is arbitrary. Thus it must be

$$(\rho \vec{v})_t + \text{div}_e(\rho \vec{v} \otimes \vec{v}) = \text{div}_e \left(\frac{1}{J} (\mathcal{T}^{(1)} \cdot G^T \hat{n}_e) \right). \quad (9.6)$$

Comparing this with the equation for the conservation of momentum in Eulerian coordinates (e.g.: see the answer to the problem in §6) we conclude that $J \boldsymbol{\tau} = \mathcal{T}^{(1)} G^T$ — this reproduces (9.4).

Note that obtaining (9.6) from (9.2) by a direct change of variables calculation ($\vec{s} \rightarrow \vec{x}$) is a rather cumbersome, and error prone, calculation. It is generally true that calculations that can be done at the integral level tend to be shorter and more elegant. This is one of the reasons why variational formulations are advantageous whenever possible.

9.2.1 Forces and stress tensors

In the Eulerian frame the forces across an arbitrary surface within the solid, are given by (see definition 1.2)

$$d\vec{f}_e = \boldsymbol{\tau} \cdot \hat{n}_e \, dA_e. \quad (9.7)$$

On the other hand, from the third formula in #3 above, and

#1, we have $\frac{d}{dt} \int_{\Omega_e} \rho \vec{v} \, dV_e = \int_{\partial\Omega_l} \mathcal{T}^{(1)} \cdot \hat{n}_l \, dA_l$. Thus

$$d\vec{f}_e = \mathcal{T}^{(1)} \cdot \hat{n}_l \, dA_l. \quad (9.8)$$

This also follows by substituting (9.12b-c) into (9.7) and using

(9.4). Finally, if we introduce the *pull-back* of the force elements to the reference frame, $\mathbf{d}\vec{f}_l = \mathbf{H}\mathbf{d}\vec{f}_e$, we have

$$\mathbf{d}\vec{f}_l = \mathcal{T}^{(2)} \cdot \hat{\mathbf{n}}_l \, \mathbf{d}A_l, \quad (9.9)$$

where

$$\mathcal{T}^{(2)} = \mathbf{H} \mathcal{T}^{(1)} = \mathbf{J} \mathbf{H} \boldsymbol{\tau} \mathbf{H}^T = \rho_0 \mathbf{M} \quad (9.10)$$

is the **second Piola-Kirchhoff stress tensor** (it is **symmetric**).

9.3 A few useful formulas

So as not to interfere with the continuity of the exposition in this section, we list (and prove) here some formulas that are needed. Recall that

$$\vec{\mathbf{x}} = \vec{\mathbf{X}}(\vec{\mathbf{s}}, t), \quad \vec{\mathbf{s}} = \vec{\mathbf{S}}(\vec{\mathbf{x}}, t), \quad \delta\vec{\mathbf{x}} = \mathbf{G} \delta\vec{\mathbf{s}}, \quad \delta\vec{\mathbf{s}} = \mathbf{H} \delta\vec{\mathbf{x}}, \quad \mathbf{H} = \mathbf{G}^{-1}, \quad G_{ij} = \frac{\partial x_i}{\partial s_j}, \quad \text{and} \quad H_{ij} = \frac{\partial s_i}{\partial x_j}.$$

Furthermore: note that **we use the repeated index summation convention**.

1. We have
$$\frac{\partial \sigma_{pq}}{\partial G_{nm}} = \frac{1}{2} (\delta_{pm} G_{nq} + \delta_{qm} G_{np}) \quad \text{and} \quad \frac{\partial H_{nm}}{\partial G_{ij}} = -H_{ni} H_{jm}. \quad (9.11)$$

Proofs:

(A) For any parameter: $\sigma_\zeta = \frac{1}{2}(G_\zeta^T G + G^T G_\zeta)$. Now use this with $\zeta = G_{nm}$, so that $(G_\zeta)_{ij} = \delta_{in} \delta_{jm}$.

(B) Let $\zeta = G_{ij}$. Then ∂_ζ of $\delta_{nk} = H_{nl} G_{lk}$ yields: $0 = (H_\zeta)_{nl} G_{lk} + H_{ni} \delta_{kj}$. Now multiply by H_{km} . ♣

2. **On surfaces: how tangents, normals, and areas transform.** Consider a surface in the Lagrangian frame, \mathcal{S}_l , with its corresponding surface in the Eulerian frame, \mathcal{S}_e . Consider now: (i) $\hat{\mathbf{t}}_l$, a tangent vector at some point in \mathcal{S}_l . (ii) $\hat{\mathbf{n}}_l$, a normal vector at some point in \mathcal{S}_l . (iii) $\mathbf{d}A_l$, the area element on \mathcal{S}_l . Furthermore, let $\hat{\mathbf{t}}_e$, $\hat{\mathbf{n}}_e$, and $\mathbf{d}A_e$ be the corresponding objects on \mathcal{S}_e . Then †

$$[\mathbf{a}] \hat{\mathbf{t}}_e = \mathbf{a} \mathbf{G} \hat{\mathbf{t}}_l, \quad [\mathbf{b}] \hat{\mathbf{n}}_e = \mathbf{b} \mathbf{H}^T \hat{\mathbf{n}}_l, \quad \text{and} \quad [\mathbf{c}] \mathbf{b} \, \mathbf{d}A_e = \mathbf{J} \, \mathbf{d}A_l, \quad (9.12)$$

where: $\mathbf{a} = \mathbf{1}/\|\mathbf{G} \hat{\mathbf{t}}_l\| = \|\mathbf{H} \hat{\mathbf{t}}_e\|$ and $\mathbf{b} = \mathbf{1}/\|\mathbf{H}^T \hat{\mathbf{n}}_l\| = \|\mathbf{G}^T \hat{\mathbf{n}}_e\|$.

† Note that here we consider the map between surfaces at a given arbitrary, but fixed, time.

Proofs: **[a]** Consider a curve in \mathcal{S}_l , $\vec{\mathbf{s}} = \vec{\mathbf{s}}(\mu)$. The corresponding curve in \mathcal{S}_e is $\vec{\mathbf{x}} = \vec{\mathbf{x}}(\mu) = \vec{\mathbf{X}}(\vec{\mathbf{s}}(\mu), t)$. Then take the derivative with respect to μ of this last formula, and normalize to unit length.

[b] Any smooth surface can be realized, at least locally, as a level set. Thus let $\phi_e(\vec{\mathbf{x}}) = 0$ define \mathcal{S}_e . Thus $0 = \phi_l(\vec{\mathbf{s}}) = \phi_e(\vec{\mathbf{X}}(\vec{\mathbf{s}}, t))$ for \mathcal{S}_l . Hence $\nabla_l \phi_l = G^T \nabla_e \phi_e$, or $H \nabla_l \phi_l = \nabla_e \phi_e$. Now normalize to unit length.

[c] Consider the surface \mathcal{S}_l , and the nearby surface \mathcal{S}_l^n defined by: *Displace \mathcal{S}_l by the amount $\hat{\mathbf{n}}_l \, dr_l$ at every point along the surface, for some “infinitesimal” dr_l (dr_l need not be constant, but is smooth).* Since the map $\vec{\mathbf{s}} \rightarrow \vec{\mathbf{x}}$ takes $\vec{\mathbf{s}} + \hat{\mathbf{n}}_l \, dr_l$ to $\vec{\mathbf{x}} + (\mathbf{G} \hat{\mathbf{n}}_l) \, dr_l$, we see that \mathcal{S}_l^n gets mapped to \mathcal{S}_e^n — which is defined relative to \mathcal{S}_e in the same way as \mathcal{S}_l , using $dr_e = (\hat{\mathbf{n}}_e \cdot \mathbf{G} \hat{\mathbf{n}}_l) \, dr_l$. Hence using **[b]** we see that $dr_e = b \, dr_l$. Now consider the volume between a surface element in \mathcal{S}_l and \mathcal{S}_l^n ; which is given by $dV_l = \mathbf{d}A_l \, dr_l$. The corresponding volume, attached to \mathcal{S}_e , is $dV_e = \mathbf{d}A_e \, dr_e$. Now, using $dr_e = b \, dr_l$ and $dV_e = \mathbf{J} \, dV_l$, **[c]** follows. ♣

The End.