18.376 - The Laplace Transform
Examples of its use for pde

R. R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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Theme: Laplace Transform (LT) for Initial Value Problem (IVP). (1) Solve using Green's functions. (2) Use inverse LT and residues to get solution in terms of normal modes.

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1 Heat equation in an interval, with Dirichlet BC (example 1)

Consider the heat equation in an interval
$$u_t - u_{xx} = 0, \quad 0 < x < 1 \quad \text{and} \quad t > 0, \tag{1.1}$$
with boundary conditions $u(0,\, t) = u(1,\, t) = 0$ and initial condition $u(x,\, 0) = f(x)$. Let
$$U = U(x,\, s) = \int_0^\infty e^{-s\,t} u(x,\, t)\, dt \tag{1.2}$$
be the Laplace transform in time of u. Then

1. Write the equation that U satisfies. *It is a forced (by f) ode problem in $0 < x < 1$, for every s, with some boundary conditions at $x = 0,\, 1$.*
2. Find the Green's function for the problem, so that you can write
$$U(x,\, s) = \int_0^1 G(x,\, y,\, s)\, f(y)\, dy. \tag{1.3}$$
3. G may have no branch points, but it has infinitely many poles. Find these poles.

   Note: even though $s = 0$ appears to be a singularity of G, it is not one.
4. Compute the residue r_n of G at the pole s_n.
5. Note that the residues are functions of x and y, $r_n = r_n(x,\, y)$. Show now that
$$u = \sum_{n=1}^{\infty} \int_0^1 r_n(x,\, y)\, f(y)\, dy\, e^{s_n\, t} \tag{1.4}$$
   is the same formula for the solution of (1.1) as the one that results from normal modes.

Where/how does (1.4) arise? Recall the formula for the inverse Laplace transform
$$u = \frac{1}{2\,\pi\, i} \int_\Gamma U(x,\, s)\, e^{s\, t}\, ds, \tag{1.5}$$
where Γ is a path in the complex plane of the form $s = a + i\,\mu$, with $-\infty < \mu < \infty$ and $a > 0$ large enough. If this path is "moved" to the left ($a \to -\infty$), every time it crosses a pole of U it picks up the residue of $U\, e^{s\, t}$ there. Because of (1.3), these residues are precisely the terms in the sum in (1.4).

Warning. *The reduction of (1.5) to a sum over residues is not always possible. For this the function U must have pole singularities only, and (for example) vanish as $Re(s) \to -\infty$. This is not always true. For example:*[1] *sometimes U is not even defined for $Re(s) <$ some constant, or it may have singularities other than poles, or it may not behave properly as $Re(s) \to -\infty$.*

Answer: Heat equation in an interval, with Dirichlet BC (example 1)

1. From the definition of U, we have $\quad s\,U - U_{xx} = f, \quad 0 < x < 1, \quad$ (1.6), with $U(0, s) = U(1, s) = 0$.

2. The Green's function for (1.5) satisfies the equation and boundary conditions for $f = \delta(x - y)$. Thus it has the form

$$G = \beta_1(y)\,\sinh(\sqrt{s}\,x) \;\text{ for }\; x < y \quad\text{and}\quad G = \beta_2(y)\,\sinh(\sqrt{s}\,(x-1)) \;\text{ for }\; x > y,$$

for some functions β_1 and β_2. In addition,
(a) G is continuous at $x = y$ .... $\beta_1(y)\,\sinh(\sqrt{s}\,y) = \beta_2(y)\,\sinh(\sqrt{s}\,(y-1))$.
(b) G_x "jumps" by -1 as x crosses $x = y$,
so that $-G_{xx}$ produces the δ function
on the right. That is .... $\sqrt{s}\,\beta_1(y)\,\cosh(\sqrt{s}\,y) - \sqrt{s}\,\beta_2(y)\,\cosh(\sqrt{s}\,(y-1)) = 1$

These two conditions determine β_1 and β_2, and we get

$$G = \frac{-1}{\sqrt{s}\,\sinh(\sqrt{s})}\,\sinh(\sqrt{s}\,x)\,\sinh(\sqrt{s}\,(y-1)) \quad\text{for}\quad x < y. \tag{1.7}$$

$$G = \frac{-1}{\sqrt{s}\,\sinh(\sqrt{s})}\,\sinh(\sqrt{s}\,y)\,\sinh(\sqrt{s}\,(x-1)) \quad\text{for}\quad y < x. \tag{1.8}$$

Equation (1.3) applies with G given as above. Furthermore: from these formulas it should be obvious that G *does not depend on which root $\sqrt{s}$ is selected.* Changing $\sqrt{s} \to -\sqrt{s}$ does not alter G. Thus G *is single valued, with no branch points.*

3. Clearly, singularities of G can occur only at the places where $\sqrt{s}\,\sinh(\sqrt{s})$ vanishes, that is, at:[2] $s = 0$ and $s = s_n = -n^2\,\pi^2$, where $n = 1, 2, 3, \ldots$ In fact, by expanding near $s = 0$ both the numerators and denominators, we obtain

$$\begin{aligned} G &= \frac{-1}{s + \frac{1}{6}\,s^2 + \ldots}\left(s\,x\,(y-1) + \frac{1}{6}\,\left(x\,(y-1)^3 + x^3\,(y-1)\right)\,s^2 + \ldots\right) \\ &= -x\,(y-1) + O(s) \quad\text{for}\;\; x < y \;\text{ and } s \text{ small.} \end{aligned} \tag{1.9}$$

$$G = -y\,(x-1) + O(s) \quad\text{for}\;\; y < x \;\text{ and } s \text{ small.} \tag{1.10}$$

Thus, **$s = 0$ is not a singularity of G**. Similarly, expanding for s close to s_n, we obtain

$$\begin{aligned} \sqrt{s}\,\sinh(\sqrt{s}) &= \tfrac{1}{2}\,\cosh(i\,n\,\pi)\,(s - s_n) + O\left((s - s_n)^2\right) \\ &= \tfrac{1}{2}\,(-1)^n\,(s - s_n) + O\left((s - s_n)^2\right), \\ \sinh(\sqrt{s}\,x)\,\sinh(\sqrt{s}\,(y-1)) &= -\sin(n\,\pi\,x)\,\sin(n\,\pi\,(y-1)) + O(s - s_n) \\ &= (-1)^{n+1}\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,y) + O(s - s_n), \\ \sinh(\sqrt{s}\,y)\,\sinh(\sqrt{s}\,(x-1)) &= (-1)^{n+1}\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,y) + O(s - s_n). \end{aligned}$$

Thus

$$G = \frac{2\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,y)}{(s - s_n)} + O(1) \tag{1.11}$$

for $s - s_n$ small, and both $x < y$ or $y < x$.

[1] All of these things occur quite often in problems that arise in applications.
[2] Note that $\sqrt{s_n} = \pm i\,n\,\pi$.

4. From (1.11) it follows that s_n is a simple pole of G, with residue

$$r_n = 2\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,y). \tag{1.12}$$

5. Substituting (1.12) into (1.4) yields

$$u = \sum_{n=1}^{\infty} 2\left(\int_0^1 \sin(n\,\pi\,y)\,f(y)\,dy\right)\,\sin(n\,\pi\,x)\,e^{-n^2\,\pi^2\,t}, \tag{1.13}$$

which is precisely the same as the solution by normal modes to the problem in (1.1).

Why is it that are the poles the two expressions for G — i.e.: for $x > y$ and $x < y$ — become the same? The singularities of G occur at the values of s for which there is no solution of the equation for G. That is, the places where the equations determining β_1 and β_2 are singular. But these are exactly the values of s at which the functions $\sinh(\sqrt{s}\,x)$ and $\sinh(\sqrt{s}\,(x-1))$ are proportional to each other.

2 Wave equation in an interval, with Dirichlet BC (example 2)

Consider the wave equation in an interval $\quad u_{tt} - u_{xx} = 0, \quad 0 < x < 1 \quad \text{and} \quad t > 0, \qquad (2.1)$

with boundary conditions: $\boldsymbol{u(0,\,t) = u(1,\,t) = 0}$; and initial conditions: $\boldsymbol{u = 0}$ and $\boldsymbol{u_t = f(x)}$ at $\boldsymbol{t = 0}$. Let

$$U = U(x,\,s) = \int_0^{\infty} e^{-s\,t}u(x,\,t)\,dt \tag{2.2}$$

be the Laplace transform in time of u. Now

1. Write the equation that U satisfies.

2. Find the Green's function for the problem for U, so that you can write

$$U(x,\,s) = \int_0^1 G(x,\,y,\,s)\,f(y)\,dy. \tag{2.3}$$

3. G has infinitely many poles. **Find these poles.**

Note: even though $s = 0$ appears to be a singularity of G, it is <u>not one</u>.

4. Let $r_{\pm n}$ be the residue of G at the pole $s_{\pm n}$. **Compute these residues.**

5. Note that the residues are functions of x and y, $r_{\pm n} = r_{\pm n}(x,\,y)$. **Show that**

$$u = \sum_{n=1}^{\infty} \int_0^1 \left(r_n(x,\,y)\,e^{s_n\,t} + r_{-n}(x,\,y)\,e^{s_{-n}\,t}\right)\,f(y)\,dy \tag{2.4}$$

is the same formula for the solution of (2.1) as the one that results from normal modes.

6. Use the results above to **write the formula equivalent to (2.4)**, but for the case when u satisfies the initial conditions: $\boldsymbol{u = g(x)}$ and $\boldsymbol{u_t = 0}$ at $\boldsymbol{t = 0}$.

Where/how does (2.4) arise? Recall the formula for the *inverse Laplace transform*

$$u = \frac{1}{2\,\pi\,i}\int_{\Gamma_a} U(x,\,s)\,e^{s\,t}\,ds, \tag{2.5}$$

where Γ_a is a path in the complex plane of the form $s = a + i\,\mu$, with $-\infty < \mu < \infty$ and $a > 0$ (fixed) large enough. If this path is "moved" to the left ($a \to -\infty$), every time it crosses a pole of U it picks up the residue of $U\,e^{s\,t}$ there. Because of (2.3), **these residues are precisely the terms in the sum in (2.4).**

Warning. *The reduction of (2.5) to a sum over residues is not always possible. For this U must have pole singularities only, and (for example) vanish as $Re(s) \to -\infty$. This is not always true. Sometimes U is not even defined for $Re(s) <$ some constant, or it has singularities other than poles, or it does not behave properly as $Re(s) \to -\infty$, or ...* (all these things occur in problems that arise in applications).

Answer: Wave equation in an interval, with Dirichlet BC (example 2)

1. From the definition of U, we have $$s^2\,U - U_{xx} = f, \quad 0 < x < 1, \qquad (2.6),$$ with $U(0,s) = U(1,s) = 0$.

2. The Green's function for (2.6) satisfies the ode and boundary conditions, with $f = \delta(x-y)$. Thus

$$G = \beta_1(y)\,\sinh(s\,x) \ \text{ for } \ x < y \quad \text{and} \quad G = \beta_2(y)\,\sinh(s\,(x-1)) \ \text{ for } \ x > y,$$

for some functions β_1 and β_2. In addition,
(a) G is continuous at $x = y$ .... $\beta_1(y)\,\sinh(s\,y) = \beta_2(y)\,\sinh(s\,(y-1))$.
(b) G_x "jumps" by -1 as x crosses $x = y$,
so that $-G_{xx}$ produces the δ function
on the right. That is .... $s\,\beta_1(y)\,\cosh(s\,y) - s\,\beta_2(y)\,\cosh(s\,(y-1)) = 1$

These two conditions determine β_1 and β_2, and we get

$$G \;=\; \frac{-1}{s\,\sinh(s)}\,\sinh(s\,x)\,\sinh(s\,(y-1)) \quad \text{for} \quad x < y. \qquad (2.7)$$

$$G \;=\; \frac{-1}{s\,\sinh(s)}\,\sinh(s\,y)\,\sinh(s\,(x-1)) \quad \text{for} \quad y < x. \qquad (2.8)$$

Equation (2.3) applies with G given as above.

Note that, for each $(x,\,y)$, G above is the ratio of two entire functions of s, with denominator

$$W = s\,\sinh(s) = s\,\sinh(s\,x)\,\cosh(s(x-1)) - s\,\cosh(s\,x)\,\sinh(s(x-1)) \qquad (2.9)$$

equal to the Wronskian of the solutions $\tilde{U}_1 = \sinh(s\,x)$ and $\tilde{U}_2 = \sinh(s\,(x-1))$, each satisfying one of the two boundary conditions U must satisfy (the Wronskian of two solutions is constant).

3. Clearly, singularities of G (in fact, poles) can occur only at the places where $s\,\sinh(s)$ vanishes, that is, at: $s = 0$ and $s = s_{\pm n} = \pm i\,n\,\pi$, where $n = 1,\,2,\,3,\ldots$ In fact, by expanding near $s = 0$ both the numerators and denominators, we can see that

$$G = -x\,(y-1) + O(s^2) \ \text{ for } \ x < y \quad \text{and} \quad G = -y\,(x-1) + O(s^2) \ \text{ for } \ x > y. \qquad (2.10)$$

Thus, $\boldsymbol{s = 0}$ **is not a singularity of** $\boldsymbol{G}$.

Similarly, expanding for s close to $s = s_{\pm n} = \pm i\,n\,\pi$, it is easy to see that

$$G = \pm\frac{\sin(n\,\pi\,x)\,\sin(n\,\pi\,y)}{i\,n\,\pi\,(s - s_n)} + O(1) \qquad (2.11)$$

for $s - s_n$ small. Hence **these points are, actually, poles.**

Note: the leading order in (2.11) is the same for both $x < y$ and $y < x$. Why? This is because the singularities of G occur at the values of s for which there is no solution to the equations determining G. That is, where the equations for β_1 and β_2 are singular. But these are the values of s at which $\sinh(s\,x)$ and $\sinh(s\,(x-1))$ are proportional to each other: where $W = 0$. Thus, for s very close to $s_{\pm n}$, G should be (approximately) proportional to the same function of x for both $x < y$ and $x > y$.

4. From (2.11) it follows that $s_{\pm n} = \pm i\,n\,\pi$ is a simple pole of G, with residue

$$r_{\pm n} = \pm\,\frac{1}{i\,n\,\pi}\,\sin(n\,\pi\,x)\,\sin(n\,\pi\,y). \qquad (2.12)$$

5. Substituting (2.12) into (2.4) yields

$$u = \sum_{n=1}^{\infty} \underbrace{\left(2\int_0^1 \sin(n\,\pi\,y)\,f(y)\,dy\right)}_{f_n}\,\sin(n\,\pi\,x)\,\frac{\sin(n\,\pi\,t)}{n\,\pi}, \qquad (2.13)$$

which is precisely the same as the solution by normal modes to the problem in (2.1) — written in terms of the sine-Fourier coefficients f_n for the initial data f.

6. The case with initial data $u = g(x)$ and $u_t = 0$ can be treated in a similar way to the one above. However, let $\boldsymbol{v(x,\, t) = \int_0^t u(x,\, \tau)\, d\tau}$. Then

— $v(0,\, t) = v(1,\, t) = 0$.

— $v(x,\, 0) = 0$ and $v_t(x,\, 0) = g(x)$.

— $v_{xx} = \int_0^t u_{xx}(x,\, \tau)\, d\tau = \int_0^t u_{tt}(x,\, \tau)\, d\tau = u_t(x,\, t) = v_{tt}$.

It follows that v satisfies a problem like the one in (2.1). Thus

$$v = \sum_{n=1}^{\infty} g_n\, \sin(n\,\pi\, x)\, \frac{\sin(n\,\pi\, t)}{n\,\pi} \quad \Longrightarrow \quad u = \sum_{n=1}^{\infty} g_n\, \sin(n\,\pi\, x)\, \cos(n\,\pi\, t), \tag{2.14}$$

where $g_n = 2 \displaystyle\int_0^1 \sin(n\,\pi\, y)\, g(y)\, dy$.

3 Wave equation in an interval: Dirichlet/Radiation BC (example 3)

Consider the small transversal vibrations of a semi-infinite string under tension, with the motion restricted to a plane, and the string end tied. Assume a string that consists of two homogeneous pieces, seamlessly joined. In appropriately selected a-dimensional variables, the string is described by the equations

$$\begin{aligned} 0 &= u(0,\, t), && \text{for} \quad t \geq 0, \\ 0 &= u_{tt} - u_{xx}, && \text{for} \quad t \geq 0 \ \text{ and } \ 0 < x < 1, \\ 0 &= u_{tt} - c^2\, u_{xx}, && \text{for} \quad t \geq 0 \ \text{ and } \ 1 < x, \end{aligned}$$

where $\boldsymbol{0 < c < 1}$ **is a constant,**[3] and *both u and u_x are continuous at $x = 1$.*

Let us now assume that the *initial conditions* for the string are such that *they include no perturbations to the string heavier piece on $\boldsymbol{x > 1}$.* In fact, we will assume that *no perturbations ever arise from the region $\boldsymbol{x > 1}$. This does not mean that u vanishes for $x > 1$.* In fact, what it means is:

— For $x > 1$, $u = u_R(x - c\,t)$, for some wave u_R generated from the region $0 < x < 1$.
There is no left moving component:[4] $u_L(x + c\,t) \equiv 0$.

— For $0 < x < 1$, $u = u_r(x - t) + u_l(x + t)$, for some functions u_l and u_r. At $x = 1$ the incoming right moving wave u_r produces a reflected left moving wave u_l, and a transmitted right moving wave u_R.
There is never an incoming left moving wave, u_L, from the right $(x > 1)$.

The relationship between u_r, u_l, and u_R, can be calculated using the continuity of u and u_x. However, we do not need it right now.

Because of the assumptions above, the problem can be reduced to an equation for the string in the region $0 < x < 1$, without the need to solve in the region $x > 1$. **The reduced problem is:** **(3.1)**

a. Equation: $0 = u_{tt} - u_{xx}$, for $0 < x < 1$ and $t > 0$.

b. Left boundary condition: $0 = u(0,\, t)$.

c. Right boundary condition: $0 = u_t(1,\, t) + c\, u_x(1,\, t)$.

d. Initial data: $u(x,\, 0) = \alpha(x)$ and $u_t(x,\, 0) = \beta(x)$, for some α and β.

[3] Assume that the $x > 1$ string piece is the heavier/denser one. This yields $c < 1$.

[4] The general solution (on $x > 1$) must have the form: $u = u_R(x - c\,t) + u_L(x + c\,t)$, for some functions u_L and u_R.

e. Assume α and β are smooth, and consistent with the BC: $0 = \alpha(0) = \beta(0) = \beta(1) + c\,\alpha'(1)$ (where $' = \frac{d}{dx}$). In fact, *assume that the initial perturbation is fully contained* <u>*within*</u> $0 \le x < 1$.
Thus: **Assume that α, β, and all their derivatives, vanish at $x = 1$.** **(3.1e)**

About item c: Immediately to the right of $x = 1$ the solution should have the form $u = u_R(x - c\,t)$. Because of the continuity of u and u_x at $x = 1$, this yields $u_t + c\,u_x = 0$ at $x = 1$. This condition is exactly equivalent to the statement that no disturbances from $x \ge 1$ reach into $x < 1$.

The reduced problem (3.1) gives rise to a strange normal modes problem, [5] **where the eigenvalue shows up in both the equation** <u>**and**</u> **the boundary conditions.** While this eigenvalue problem is neither self-adjoint, nor normal (at least, not in any obvious form), it turns out that **the normal modes are "OK", and that the solutions can be expanded in terms of them.** **The purpose of this problem is to show this.**

Let

$$U = U(x,\, s) = \int_0^\infty e^{-s\,t} u(x,\, t)\, dt \tag{3.2}$$

be the Laplace transform in time of u. Then

1. **Write the equation that U satisfies.**
   It is a forced ode in $0 < x < 1$, for every s, with boundary conditions at $x = 0,\, 1$. The forcing term is given by $\boldsymbol{\gamma = \gamma(x,\, s) = \beta(x) + s\,\alpha(x)}$.
   Note: (3.1e) yields $\mathbf{u(1,\, 0) = 0}$. *This plays a role simplifying the BC for U at $x = 1$.*
2. **Find the Green's function of the problem for U,** so that you can write
$$U(x,\, s) = \int_0^1 G(x,\, y,\, s)\, \gamma(y, s)\, dy. \tag{3.3}$$
3. G has infinitely many poles. **Find them.**
   Note: even though $s = 0$ appears to be a singularity of G, it is <u>not one</u>. **Show this.**
4. Let r_n be the **residue of G at the pole s_n.** **Compute these residues.**
5. Note that the residues are functions of x and y, $r_n = r_n(x,\, y)$. **Show that:**
$$u = \sum_{n=-\infty}^{\infty} \int_0^1 r_n(x,\, y)\, \gamma(y,\, s_n)\, dy\, e^{s_n\, t} \tag{3.4}$$
   provides a normal mode expansion for the solution to (3.1). That is, (3.4) has the form
$$u = \sum u_n\, \phi_n(x)\, e^{\lambda_n\, t}, \tag{3.5}$$
   where the $\phi_n(x)\, e^{s_n\, t}$ are normal modes, and the coefficients u_n are determined by the initial data *(via formulas that you should obtain).*
   Recall that the normal modes satisfy the equation and the BC, but initial conditions are not imposed on them. The initial conditions are obtained by doing linear combinations of normal modes, as in (3.5).

Where/how does (3.4) arise? Recall the formula for the *inverse Laplace transform*

$$u = \frac{1}{2\,\pi\, i} \int_{\Gamma_a} U(x,\, s)\, e^{s\,t}\, ds, \tag{3.6}$$

where Γ_a is a path in the complex plane of the form $s = a + i\,\mu$, with $-\infty < \mu < \infty$ and $a > 0$ (fixed) large enough. If this path is "moved" to the left ($a \to -\infty$), every time it crosses a pole of U it picks up the residue of $U\, e^{s\,t}$ there. Because of (3.3), **these residues are precisely the terms in the sum in (3.4).**

Warning. *The reduction of (3.6) to a sum over residues is not always possible. For this U must have pole singularities only, and (for example) vanish as $Re(s) \to -\infty$. This is not always true. Sometimes U is not even defined for $Re(s) <$ some constant, or it has singularities other than poles, or it does not behave properly as $Re(s) \to -\infty$, or ...* (all these things occur in problems arising in applications, but not here).

[5] Seek solutions of the form $\boldsymbol{u = \phi(x)\, e^{\lambda\, t}}$.

Answer: Wave equation in an interval: Dirichlet/Radiation BC (example 3)

1. From the definition of U, we have $$s^2\, U - U_{xx} = \beta + s\,\alpha = \gamma, \quad 0 < x < 1, \tag{3.7}$$
with $0 = U(0, s)$
and $0 = s\, U(1, s) + c\, U_x(1, s)$ — this second BC is equivalent to (3.1c).

Taking the Laplace Transform of the BC in (3.1c) leads, in general, to $\boldsymbol{s\,U(1,s)+c\,U_x(1,s) = u(1,\,0)}$. However, because of (3.1e), $\boldsymbol{u(1,\,0) = \alpha(1) = 0}$ — which simplifies the BC.

2. The Green's function for (3.7) satisfies the ode and boundary conditions, with $\gamma = \delta(x-y)$. Thus

$$G = \beta_1(y)\,\sinh(s\,x) \;\text{ for }\; x < y \quad \text{and} \quad G = \beta_2(y)\,J(x,\,s) \;\text{ for }\; x > y,$$

for some functions β_1 and β_2, where $$\boldsymbol{J = \tfrac{1+c}{2}\, e^{-s\,(x-1)} - \tfrac{1-c}{2}\, e^{s\,(x-1)}}. \tag{3.8}$$
Note: J satisfies the right BC. Further,
(a) G is continuous at $x = y$ .... $\beta_1(y)\,\sinh(s\,y) = \beta_2(y)\,J(y,\,s)$.
(b) G_x "jumps" by -1 as x crosses $x = y$,
so that $-G_{xx}$ produces the δ function
on the right. That is .... $s\,\beta_1(y)\,\cosh(s\,y) - \beta_2(y)\,J_x(y,\,s) = 1$.

These two conditions determine β_1 and β_2, so that

$$G(s,\,x,\,y) = \frac{1}{s\,(c\,\cosh(s) + \sinh(s))}\begin{cases} \sinh(s\,x)\,J(y,\,s) & \text{for} \quad 0 \le x \le y \le 1, \\ \sinh(s\,y)\,J(x,\,s) & \text{for} \quad 0 \le y \le x \le 1. \end{cases} \tag{3.9}$$

Equation (3.3) applies with G given as above.

Note that, for each $(x,\,y)$, G above is the ratio of two entire functions of s, with denominator [6]

$$W = s\,(c\,\cosh(s) + \sinh(s)) = J(x,\,s)\,(\sinh(s\,x))' - (J(x,\,s))'\,\sinh(s\,x) \tag{3.10}$$

equal to the Wronskian of the solutions $\tilde{U}_1 = J(x,\,s)$ and $\tilde{U}_2 = \sinh(s\,x)$, each satisfying one of the two boundary conditions that U must satisfy (the Wronskian of two solutions is constant).

3. Clearly, singularities of G (in fact, poles) can occur only at places where $W(s)$ vanishes, that is, at:

$$s_n = \frac{1}{2}\,\ln\left(\frac{1-c}{1+c}\right) + i\,n\,\pi = -\nu + i\,n\,\pi, \quad \text{where } \; n \in \mathcal{Z} \tag{3.11}$$

and $\boldsymbol{0 < \nu}$ **is defined by the formula.** This follows because $W(s) = 0$ and $s \neq 0$ if and only if $\tanh(s) = -c$, which is easily seen to be equivalent to $(1+c)\,e^{2\,s} = 1 - c$. Note that we have excluded the root $\boldsymbol{s = 0}$. The reason is that $\boldsymbol{s = 0}$ **is not a singularity of** $\boldsymbol{G}$**,** because it is also a root of the numerator of G. In fact, by expanding near $s = 0$ both the numerator and denominator of G, we see that

$$G = x + O(s) \;\text{ for }\; x < y \quad \text{and} \quad G = y + O(s) \;\text{ for }\; x > y.$$

On the other hand, expanding for s close to $s = s_n$, it is easy to see that [7]

$$G = -\frac{\sinh(s_n\,x)\,\sinh(s_n\,y)}{s_n\,(s - s_n)} + O(1) \tag{3.12}$$

for $s - s_n$ small. Hence **these points are, actually, poles.**
Note: the leading order in (3.12) is the same for both $x < y$ and $y < x$. Why? This is because the singularities of G occur at the values of s for which there is no solution to the equations determining G. That is, where the equations for β_1 and β_2 are singular. But these are the values of s at which $\sinh(s\,x)$ and $J(x,\,s)$ are proportional to each other: where $W = 0$. Thus, for s very close to s_n, G should be (approximately) proportional to the same function of x for both $x < y$ and $x > y$.

[6] Here the primes indicate derivatives with respect to x.
[7] For a list of useful formulas, see remark **3.2**.

4. From (3.12) it follows that s_n (as given by (3.11)) is a simple pole of G, with residue

$$r_n = -\frac{1}{s_n}\sinh(s_n\,x)\,\sinh(s_n\,y). \tag{3.13}$$

5. Substituting (3.13) into (3.4) yields (after some manipulation)

$$u = \sum_{n=-\infty}^{n=+\infty}\left(\alpha_n + \frac{1}{s_n}\,\beta_n\right)\,e^{s_n\,t}\,\sinh(s_n\,x), \tag{3.14}$$

where

$$\alpha_n = -\int_0^1 \sinh(s_n\,x)\,\alpha(x)\,dx \quad \text{and} \quad \beta_n = -\int_0^1 \sinh(s_n\,x)\,\beta(x)\,dx. \tag{3.15}$$

Clearly: (3.14) is a sum over the normal modes of (3.1), with coefficients $u_n = \alpha_n + \frac{1}{s_n}\,\beta_n$ given by (3.15). It is easy to check that $\boldsymbol{u = e^{s_n\,t}\,\sinh(s_n\,x)}$ is a normal mode. It clearly satisfies the equation, and the BC at $\boldsymbol{x = 0}$. The BC at $\boldsymbol{x = 1}$ reduces to $\boldsymbol{s_n\,(\sinh(s_n) + c\,\cosh(s_n)) = 0}$, also satisfied. ***Because*** $\boldsymbol{Re(s_n) = -\nu < 0}$***, the normal modes decay like*** $\boldsymbol{e^{-\nu\,t}}$***, and oscillate with angular frequencies*** $\boldsymbol{\omega_n = n\,\pi}$***.***

Remark 3.1 *The expansion in (3.14–3.15) has several un-usual features, born from the fact that it arises from a non-normal problem. Two examples:*
— The eigenmodes are not orthogonal.
— The expansion for the initial data mixes the two functions. That is

$$\alpha(x) = \sum_{n=-\infty}^{n=+\infty}\left(\alpha_n + \frac{1}{s_n}\,\beta_n\right)\sinh(s_n\,x) \quad \textit{and} \quad \beta(x) = \sum_{n=-\infty}^{n=+\infty}(s_n\,\alpha_n + \beta_n)\,\sinh(s_n\,x).$$

Hence the coefficients in the expansion for α (or β) depend on <u>both</u> α and β.

Remark 3.2 Some useful formulas satisfied by the $\boldsymbol{s_n = -\nu + i\,n\,\pi,\ \ n \in \mathcal{Z}}$.

a) $\boldsymbol{\nu > 0}$ *is defined by* .... $\boldsymbol{\nu = -\frac{1}{2}\,\ln\left(\frac{1-c}{1+c}\right)}$.

b) *Thus* .... $\boldsymbol{\sinh(\nu) = c\,\cosh(\nu) = \frac{c}{\sqrt{1-c^2}}}$ *and* $\boldsymbol{e^{\nu} = \sqrt{\frac{1+c}{1-c}}}$.

c) *The $\boldsymbol{s_n}$ are the solutions to* .... $\boldsymbol{\sinh(s_n) = -c\,\cosh(s_n)}$.

d) *In particular* .... $\boldsymbol{\cosh(s_n) = \frac{(-1)^n}{\sqrt{1-c^2}}}$ *and* $\boldsymbol{e^{s_n} = (-1)^n\,\sqrt{\frac{1-c}{1+c}}}$.

e) *Since* $\boldsymbol{W = s\,(c\,\cosh(s) + \sinh(s))}$, ... $\boldsymbol{W(s_n) = 0}$.

f) *Furthermore, they are simple zeros of* $\boldsymbol{W}$, ... $\boldsymbol{\frac{dW}{ds}(s_n) = (-1)^n\,s_n\,\sqrt{1-c^2}}$.

g) *Finally,* .... $\boldsymbol{J(x,\,s_n) = (-1)^{n+1}\,\sqrt{1-c^2}\,\sinh(s_n\,x)}$.

THE END.