

Lecture 12: Black-Scholes-Merton and Beyond

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Dynamic Hedging

In the previous lecture we considered how to hedge risk for a single time period τ given the probability distribution of price, the payoff function $y(x)$, a current price x_0 and an interest rate r . All we need to do is to find the best linear fit

$$\tilde{w}_0^* = \tilde{u}_0^* + \phi^* \tilde{x}_0, \quad (1)$$

for the payoff function $y(x) = w(x, t + \tau)$ (here $\tilde{x}_0 = x_0 e^{r\tau}$, $\tilde{w}_0 = w_0 e^{r\tau}$, and t is the current moment of time).

From now on, we assume that r is risk-free (which actually is not the case). Now we are going to repeat this process for N periods of time. In particular, we are interested in those cases when we can eliminate any risk and determine $w(x, t)$ uniquely. Basically, there are only two suitable cases.

The first one is the so-called *Binomial Model* (or *Binomial Tree*). Here at each step we have only two possible outcomes of x . Thus, linear fit (1) can be found in a unique way because there is only one line passing through two points.

The second one is the Black-Scholes-Merton case. It is the continuum limit with the time step τ tending to zero, and, thus, $\delta x \rightarrow 0$. Obviously, here our best linear fit degenerates into a tangent to the payoff function which is unique (provided we have a nice payoff function). There are some concerns when we let $\tau \rightarrow 0$. For instance, in this case transaction cost grows to infinity. Another shortcoming is that the underlying asset takes independent steps after some finite correlation time; thus, we cannot assume that our process is Markovian.

Binomial Tree

Let x_+ and x_- denote two possible outcomes for the current price x . Similarly, let w_+ and w_- be the corresponding payoffs. Then the equation for the fitting line is:

$$\tilde{w}_0 = \frac{(\tilde{x}_0 - x_-)w_+ + (x_+ - \tilde{x}_0)w_-}{x_+ - x_-}, \quad (2)$$

which implies that the hedge ratio is:

$$\phi(t) = \frac{w_+ - w_-}{x_+ - x_-}.$$

(Here again $\tilde{x}_0 = x_0 e^{r\tau}$ and $\tilde{w}_0 = w_0 e^{r\tau}$.)

In the following sections we will solve the equation (2) backwards in time.

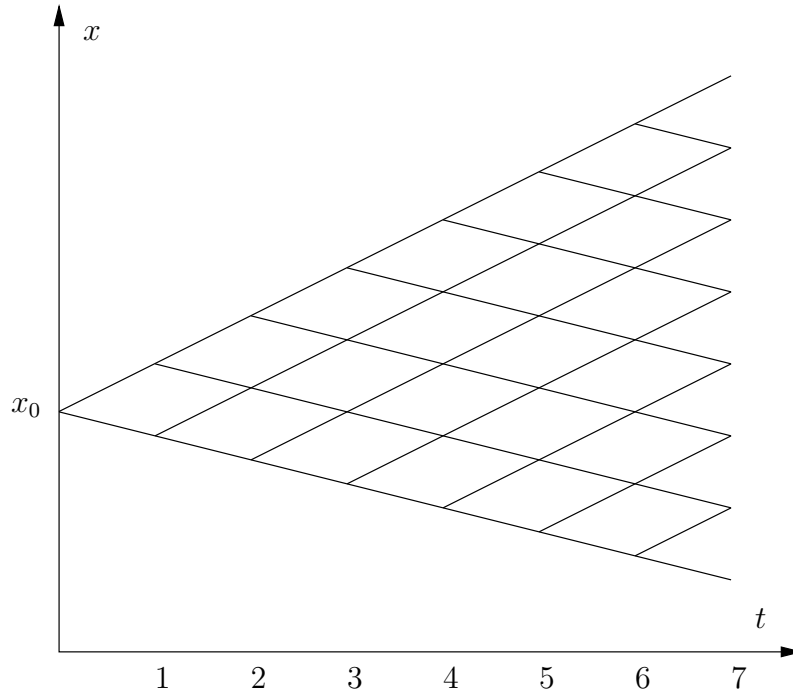


Figure 1: Binomial Tree

Continuum Limit

Normal (Additive) Stochastic Process for x_t

Let us consider the process where $x_{\pm} = x_0 \pm a$. Assuming that $r = 0$, we can rewrite the equation (2) as follows:

$$w(x, t) = \frac{1}{2} (w(x + a, t + \tau) + w(x - a, t + \tau)). \quad (3)$$

It is important to note that although the probabilities of x_{\pm} could be different, we would still have the same equation (3).

Now we can consider the process as a Bernoulli random walk with probabilities 1/2 in backward time. Transforming (3), we get:

$$w(x, t) - w(x, t + \tau) = \frac{1}{2} (w(x + a, t + \tau) - 2w(x, t + \tau) + w(x - a, t + \tau))$$

$$\begin{aligned} \left(w(x, t + \tau) - \tau \frac{\partial w}{\partial t} + \dots \right) - w(x, t + \tau) &= \frac{1}{2} \left(w(x, t + \tau) + a \frac{\partial w}{\partial x} + \frac{a^2}{2} \frac{\partial^2 w}{\partial x^2} + \dots \right. \\ &\quad \left. - 2w(x, t + \tau) + w(x, t + \tau) - a \frac{\partial w}{\partial x} + \frac{a^2}{2} \frac{\partial^2 w}{\partial x^2} + \dots \right). \end{aligned}$$

Here all partial derivatives are evaluated at the point $(x, t + \tau)$.

Now, omitting all higher-order terms (assuming that $\tau \rightarrow 0$ and $a \rightarrow 0$), we obtain the diffusion equation:

$$-\frac{\partial w}{\partial t} = D \frac{\partial^2 w}{\partial x^2}, \quad (4)$$

where

$$D = \frac{a^2}{2\tau}$$

is a diffusion coefficient.

Thus, we see that the option price ‘diffuses’ in backward time from the known payoff at $t = T$.

Lognormal (Multiplicative) Process for x_t

Here we will consider the binomial model with $x_{\pm} = xe^{\mu \pm \sigma \sqrt{\tau}}$. Expanding these outcomes we have:

$$\begin{aligned} x_{\pm} &= x \left(1 + \mu\tau \pm \sigma\sqrt{\tau} + \frac{\sigma^2\tau}{2} + \dots \right) \\ &= x (1 + \bar{\mu}\tau \pm \sigma\sqrt{\tau} + \dots), \end{aligned}$$

where $\bar{\mu} = \mu + \frac{\sigma^2}{2}$ is a noise induced drift.

Then the relative change of x is:

$$\frac{\Delta x}{x} = \Delta(\log x) = \bar{\mu}\tau \pm \sigma\sqrt{\tau}.$$

The quantities \tilde{w}_0 and \tilde{x}_0 can be represented as follows:

$$\begin{aligned} \tilde{w}_0 &= e^{r\tau} w(x, t) = (1 + r\tau)w(x, t), \\ \tilde{x}_0 &= e^{r\tau} x_0 = (1 + r\tau)x_0. \end{aligned}$$

Now we apply the binomial model for each time step τ :

$$(x_+ - x_-)\tilde{w}_0 = (\tilde{x}_0 - x_-)w_+ + (x_+ - \tilde{x}_0)w_-$$

Substituting \tilde{w}_0 , \tilde{x}_0 and x_{\pm} :

$$(2\sigma\sqrt{\tau}x)(1 + r\tau)w(x, t) = (r - \bar{\mu})\tau(w_+ - w_-) + \sigma\sqrt{\tau}x(w_+ + w_-),$$

where

$$w_{\pm} = w(x_{\pm}, t + \tau) = w(x, t + \tau) + (\bar{\mu}\tau \pm \sigma\sqrt{\tau})x \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + \frac{1}{2}\sigma^2\tau x^2 \left. \frac{\partial^2 w}{\partial x^2} \right|_{(x, t + \tau)} + \dots$$

Thus,

$$w_+ - w_- = 2\sigma\sqrt{\tau}x \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + \dots$$

and

$$w_+ + w_- = 2w(x, t + \tau) + 2\bar{\mu}\tau x \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + \sigma^2\tau x^2 \left. \frac{\partial^2 w}{\partial x^2} \right|_{(x, t + \tau)} + \dots$$

This gives us:

$$(1 + r\tau)w(x, t) = (r - \bar{\mu})\tau x \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + w(x, t + \tau) + \bar{\mu}\tau x \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + \frac{\sigma^2}{2}\tau x^2 \left. \frac{\partial^2 w}{\partial x^2} \right|_{(x, t + \tau)},$$

and the Black-Scholes miracle occurs: the drift (or expected return in x_t) drops out. However, $\bar{\mu}$ is present in higher-order terms.

Now, simplifying the obtained equation, we get:

$$\frac{w(x, t) - w(x, t + \tau)}{\tau} + rw(x, t) = rx \left. \frac{\partial w}{\partial x} \right|_{(x, t + \tau)} + \frac{\sigma^2 x^2}{2} \left. \frac{\partial^2 w}{\partial x^2} \right|_{(x, t + \tau)}.$$

Taking the limit with $\tau \rightarrow 0$, we obtain the *Black-Scholes equation*:

$$\frac{\partial w}{\partial t} + rx \frac{\partial w}{\partial x} + \frac{\sigma^2}{2} x^2 \frac{\partial^2 w}{\partial x^2} = rw. \quad (5)$$

Recall that we are assuming that $w(x, t)$ is *independent* of measure of risk expected return μ and r is a *risk-free* rate.

Risk-Neutral Valuation

Let us eliminate dimensions from the Black-Scholes equation. Namely, let us introduce the following variables:

$$\begin{aligned} \bar{t} &= \frac{T - t}{T}, \\ \bar{x} &= \log \frac{x}{k}, \\ \bar{w} &= e^{r(T-t)} w, \\ \bar{r} &= rT, \\ \bar{\sigma}^2 &= \sigma^2 T. \end{aligned}$$

Then, the partial derivatives have to be:

$$\begin{aligned} \frac{\partial}{\partial \bar{x}} &= x \frac{\partial}{\partial x}, \\ \frac{\partial}{\partial \bar{t}} &= -\frac{1}{T} \frac{\partial}{\partial t}. \end{aligned}$$

Now, equation (5) becomes:

$$\frac{\partial \bar{w}}{\partial \bar{t}} = \left(\bar{r} - \frac{\bar{\sigma}^2}{2} \right) \frac{\partial \bar{w}}{\partial \bar{x}} + \frac{\bar{\sigma}^2}{2} \frac{\partial^2 \bar{w}}{\partial \bar{x}^2}. \quad (6)$$

The Green function (solution for the initial condition $\bar{w}(\bar{x}, 0) = \delta(\bar{x})$) for (6) is:

$$\bar{G}(\bar{x}, \bar{t}) = \frac{1}{\sqrt{2\pi\bar{\sigma}^2\bar{t}}} \exp \left[-\frac{(\bar{x} + \bar{t}(\bar{r} - \bar{\sigma}^2/2))^2}{2\bar{\sigma}^2\bar{t}} \right], \quad (7)$$

which is a normal distribution with mean $\bar{t}(\bar{r} - \bar{\sigma}^2/2)$ and variance $\bar{\sigma}^2\bar{t}$. Having this, we can write a solution with the arbitrary initial condition $\bar{y}(\bar{x})$ as a convolution:

$$\bar{w}(\bar{x}, \bar{t}) = (\bar{G} \star \bar{y})(\bar{x}, \bar{t}),$$

which can be simplified as follows:

$$\bar{w}(\bar{x}, \bar{t}) = \langle \bar{y}(\bar{x}, \bar{t}) \rangle_{\bar{G}}.$$

Putting the dimensions back, we have the Green function for the Black-Scholes equation (5)

$$G(x, t) = e^{r(t-T)} L(x, t),$$

where $L(x, t)$ is a lognormal density with expected return r and volatility σ^2 which solves the following SDE for a lognormal process from t to T :

$$dx = r x dt + \sigma x dz$$

(Recall that in lecture 10 we showed that the mean of such a lognormal random variable with expected rate of return m and volatility σ has expected value $x_0 e^{(m+\sigma^2/2)(T-t)}$, and here $m = r - \sigma^2/2$.) In terms of the Green function, the general solution can be written as an expectation of the payoff with respect to the lognormal process

$$w(x, t) = e^{r(t-T)} \langle y(x) \rangle.$$

This appears to be the same as Bachelier’s fair price, equal to the expected payoff (discounted at the risk-free interest rate), except for one subtle difference: The mean rate of return μ in the lognormal process for the underlying asset has been replaced by r , the risk-free rate! This is again the “Black-Scholes miracle” caused by the hedging procedure, neglected by Bachelier, which removes any dependence on the mean return relative to the risk-free rate. Rather than solving the Black-Scholes PDE, therefore, we can instead apply the simple Bachelier “fair-game” principle, replacing μ with r . This procedure of *risk neutral valuation* is a powerful and widely used tool in options pricing and hedging.

In reality, however, a perfect hedge is not possible, and risk neutral valuation is only a first approximation. The presence of residual risk can be treated by various methods such as the Bouchaud-Sornette theory introduced in the previous lecture. In the continuum limit, this leads to perturbations of the Black-Scholes equation and corrections to risk-neutral valuation.

For more details, see the solutions to Problem Set 3 and the final project of Ken Gosier from 2001, *Derivatives Pricing and Hedging with Residual Risk*, both available online.