Lecture 8: Example of Saddle-Point Asymptotics

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1 Bernoulli ("Simple") Random Walk

To illustrate the saddle-point method that we discussed in the previous lecture, we consider the asymptotics of a Bernoulli random walk in one dimension. In the Bernoulli random walk, the individual steps are i.i.d. with probability density function given by

$$p(m) = \frac{1}{2}\left(\delta_{m,1} + \delta_{m,-1}\right), \tag{1}$$

where $\delta_{i,j}$ is the Kronecker delta and $m \in Z$. The probability density function after n steps can be written using Bachelier's equation in its discrete form:

$$P_n(m) = \sum_{j=-\infty}^{\infty} p(j) P_{n-1}(m-j). \tag{2}$$

Because this problem is discrete, we proceed with the analysis by using Fourier series (or equivalently, the z-transform) instead of Fourier transforms:

$$\hat{p}(k) = \sum_{m=-\infty}^{\infty} e^{ikm} p(m) \tag{3}$$

$$p(m) = \int_{-\pi}^{\pi} e^{-ikm} \hat{p}(k) \frac{dk}{2\pi}, \tag{4}$$

where k is a continuous variable and m is discrete. For convenience, we will continue to refer to $\hat{p}(k)$ as the transform of $p(m)$.

For the Bernoulli random walk, the characteristic function, $\hat{p}(k)$, is given by

$$\hat{p}(k) = cos(k). \tag{5}$$

As in the continous case, the transform of a convolution (in this case discrete) is the product of the transforms. Applying this result, we arrive at the usual expression for $\hat{P}_n(k)$:

$$\hat{P}_n(k) = \cos^n(k), \tag{6}$$

which immediately leads to the result

$$P_n(m) = \int_{-\pi}^{\pi} e^{-ikm} \cos^n(k) \frac{dk}{2\pi}. \tag{7}$$

This integral can be evaluated exactly using contour integration (see Supplementary Note on Lecture 16 in the course reader) to obtain

$$P_n(m) = \left[1 + (-1)^{n+m}\right] 2^{-n-1} \frac{n!}{\Gamma\left(1 + \frac{n-|m|}{2}\right)\Gamma\left(1 + \frac{n+|m|}{2}\right)}, \qquad n > |m|. \tag{8}$$

While it is possible to examine the large n limit by using Stirling's formula to approximate the factorial and gamma functions in equation (8), it is instructive to apply the saddle-point method directly to the integral (7) to find the asymptotic limit for $P_n(m)$.

We begin the analysis by making a few observations about the integrand, $f(k) = e^{-ikm}\cos^n(k)$, in equation (7):

- $f(k)$ is 2π-periodic
- $f(k+\pi) = (-1)^{n+m} f(k)$.

Using these observations, we can manipulate the integral for $P_n(m)$ into an integral over the interval $(-\pi/2, \pi/2)$:

$$\begin{aligned}
\int_{-\pi}^{\pi} e^{-ikm}\cos^n(k)\frac{dk}{2\pi} &= \int_{-\pi/2}^{3\pi/2} e^{-ikm}\cos^n(k)\frac{dk}{2\pi} \\
&= \left(\int_{-\pi/2}^{\pi/2} + \int_{\pi/2}^{3\pi/2}\right) e^{-ikm}\cos^n(k)\frac{dk}{2\pi} \\
&= \left[1 + (-1)^{n+m}\right]\int_{-\pi/2}^{\pi/2} e^{-ikm}\cos^n(k)\frac{dk}{2\pi},
\end{aligned}$$

where we used the 2π-periodicity of $f(k)$ in the first step to shift the domain of integration and the behavior of $f(k)$ under a shift of π to change the integral over $(\pi/2, 3\pi/2)$ into an integral over $(-\pi/2, \pi/2)$. Thus, we find that

$$P_n(m) = \left[1 + (-1)^{n+m}\right]\int_{-\pi/2}^{\pi/2} e^{-ikm}\cos^n(k)\frac{dk}{2\pi}. \tag{9}$$

We are now ready to use the saddle-point method. For convenience, we write $P_n(m)$ as

$$P_n(m) = \frac{\left[1 + (-1)^{n+m}\right]}{2\pi} I(n,m), \tag{10}$$

where $I(n,m)$ is defined by

$$I(n,m) = \int_{-\pi/2}^{\pi/2} e^{-ikm}\cos^n(k)\frac{dk}{2\pi}. \tag{11}$$

To apply the saddle-point method, we rewrite $I(n,m)$ as

$$I(n,m) = \int_{-\pi/2}^{\pi/2} e^{-n\psi(k,\xi)} dk, \tag{12}$$

where $\psi(k,\xi) \equiv i\xi k - \log(\cos k)$ and $\xi \equiv m/n$. Thinking of this integral as an integral along a contour in the complex k-plane, we get the best asymptotic approximation to (12) by deforming the contour so that it takes a steepest descent path through the saddle point of $\psi(k,\xi)$ that makes the value of $\left|e^{-n\psi(k,\xi)}\right|$ smallest while avoiding singularities and branch cuts. Then on this deformed contour, most of the contribution to the integral comes from the region around the saddle point, and possibly the endpoints. For this problem, it turns out that we can neglect endpoint contributions because

$$\psi(\pm\pi/2) \to \infty \tag{13}$$

so that the integrand approaches zero exponentially quickly as the contour approaches the endpoints[1].

To locate the saddle points for $\psi(k,\xi)$, we set the first derivative of $\psi(k,\xi)$ to zero:

$$0 = \left.\frac{d\psi}{dk}\right|_{k_s} = i\xi + \tan k_s. \tag{14}$$

Writing $\tan k_s$ in terms of exponentials using the substitution $\omega = e^{ik_s}$ transforms this equation into

$$\frac{1}{i}\left(\frac{\omega - \omega^{-1}}{\omega + \omega^{-1}}\right) = -i\xi \tag{15}$$

which can be solved for ω to obtain

$$\omega = \sqrt{\frac{1+\xi}{1-\xi}}. \tag{16}$$

Thus, we have

$$k_s = \frac{i}{2}\log\left(\frac{1-\xi}{1+\xi}\right). \tag{17}$$

Note that by definition $-1 \le \xi \le 1$, so the logarithm on the principal branch is real, except at $|\xi| = 1$ which must be handled separately. Thus, there are an infinite number of saddle points on the line $\mathrm{Im}k = \frac{1}{2}\log\left[(1-\xi)/(1+\xi)\right]$ located at the points with $\mathrm{Re}k = j\pi$, j an integer. It turns out that it is only possible to deform the contour in equation (12) into the saddle points with j even so that the contour is a path of steepest descent. Note that $k_s \to 0$ as $\xi \to 0$ which is the limit we took in analyzing the central region.

The last pieces of information we need are the value of ψ and $\frac{d^2\psi}{dk^2}$ at the saddle point. Plugging k_s into the definition of $\psi(k)$, we obtain

$$\begin{aligned}
\psi(k_s) &= i\xi k - \log(\cos k) \\
&= \frac{i^2\xi}{2}\log\left(\frac{1-\xi}{1+\xi}\right) - \log\sqrt{\frac{1}{1-\xi^2}} \\
&= \frac{\xi}{2}\log\left(\frac{1+\xi}{1-\xi}\right) + \frac{1}{2}\log\left(1-\xi^2\right),
\end{aligned} \tag{18}$$

where the $\cos(k_s)$ was computed by writing cosine in terms of ω and simplifying. Taking the second derivative of ψ, we find that

$$\frac{d^2\psi}{dk^2}(k) = \sec^2(k) = 1 + \tan^2(k). \tag{19}$$

Evaluating this expression at k_s, we have

$$\frac{d^2\psi}{dk^2}(k_s) = 1 + \tan^2(k) = 1 - \xi^2. \tag{20}$$

[1]Note that a more careful analysis would have been necessary to handle the endpoints if we kept the original limits of integration.

Note that $\frac{d^2\psi}{dk^2}(k_s) > 0$ away from the exceptional cases $|\xi| = 1$. Putting all the pieces together, we find that the leading order asymptotic approximation to the integral (12) is

$$I(n,m) \sim e^{-n\psi(k_s)} \int_{-\infty}^{\infty} e^{-\frac{n}{2}(1-\xi^2)(z-k_s)^2} dz = \sqrt{\frac{2\pi}{n(1-\xi^2)}} e^{-n\psi(k_s)}. \tag{21}$$

where we have chosen a horizontal path through the saddle point, k_s, so that the contour is a steepest descent path. Using this expression in the equation (10), we find that the leading approximation for the probability density function after n steps is given by

$$P_n(m) = P_n(n\xi) \sim \frac{[1+(-1)^{n+m}]}{\sqrt{2\pi n(1-\xi^2)}} e^{-n\psi(k_s)}. \tag{22}$$

To study the behavior of $P_n(m)$ in the central region, we take the limit $n \to \infty$ with m fixed so that $\xi \to 0$. Using the approximation $\log(1+\epsilon) \approx \epsilon$,

$$\begin{aligned} \psi(k_s) &\approx \frac{\xi}{2}(2\xi) + \frac{1}{2}(-\xi^2) \\ &= \frac{\xi^2}{2}. \end{aligned} \tag{23}$$

Thus, we find that

$$P_{n,CLT}(m) \sim \frac{[1+(-1)^{n+m}]}{\sqrt{2\pi n}} e^{-\frac{m^2}{2n}}, \tag{24}$$

which expresses the Central Limit Theorem in the central region.

The width of the central region can be determined by comparing the full asymptotic approximation, $P_n(m)$, to the CLT approximation, $P_{n,CLT}(m)$. Since, the true probability density function drops to zero for $m > n$, we expect the saddle point approximation to become exponentially small for sufficiently large ξ. On the other hand, the CLT approximation has tails that go out to infinity. Thus, an estimate of the width of the central region can be found by computing ξ where $P_n(n\xi)/P_{n,CLT}(n\xi)$ becomes exponentially small:

$$\frac{P_n(n\xi)}{P_{n,CLT}(n\xi)} = \frac{1}{\sqrt{1-\xi^2}} \exp\left[-n\left(\psi(k_s) - \xi^2/2\right)\right]. \tag{25}$$

Using the small ξ approximation for $\psi(k_s)$ (being careful to keep enough higher order terms), we find that

$$\psi(k_s) - \xi^2/2 = \frac{\xi^4}{12} + O(\xi^5). \tag{26}$$

Thus, we find that the expression in (25) is exponentially small when $n\xi^4 \approx 1$ which implies that

$$\xi = O(n^{-1/4}). \tag{27}$$

Thus, the CLT approximation is valid in the region $m = n\xi = O(n^{3/4})$, which is what we expect when the third moment is finite.