Lecture 7: Saddle Point Asymptotics Outside the Central Region

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In Lecture 7 we are presenting the general theory of the saddle point and steepest descent asymptotics of complex Laplace integrals. These may be applied to random walks to derive uniformly valid approximations, contrary to the previous lectures where the methods we considered gave approximations that were valid only in the central region of the distribution. This lecture is basically an overview of the method and further details on the subject may be found in the book by C. Bender and S. Orszag "Advanced Mathematical Methods for Scientists and Engineers".

1 General discussion of Laplace Integrals

For some real function $\psi(x)$, which is also analytic in general, define its Laplace Integral over the interval $[a, b]$ as:

$$I\left(n\right) = \int_a^b e^{-n\psi(x)} dx \tag{1}$$

where n is some positive parameter. We are concerned with approximating the integral in (1) as n becomes large. Since the integrand of (1) is strictly positive, as n gets large, the dominant contribution to the integral comes either from the minimum of ψ or from the endpoints (see figure 1). For the problems we are considering here, we integrate over the whole real line with x_0 being the value of x where $\psi(x)$ attains its minimum.

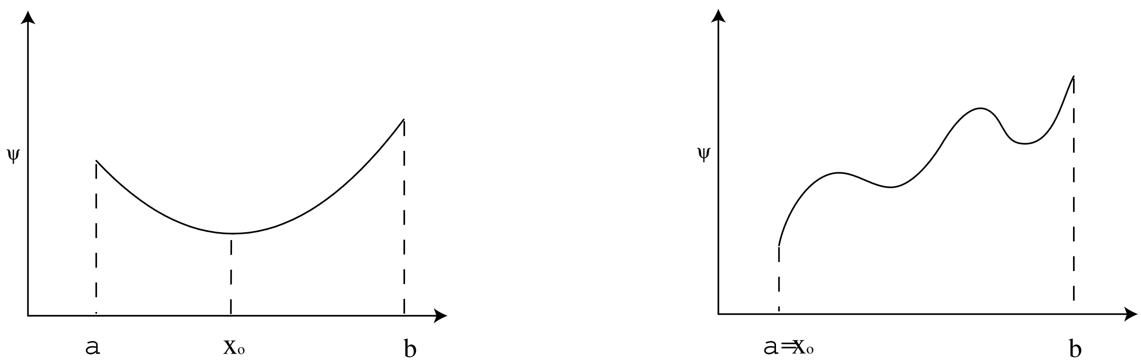

Figure 1: Examples where ψ has a minimum at the interior or at one of the endpoints

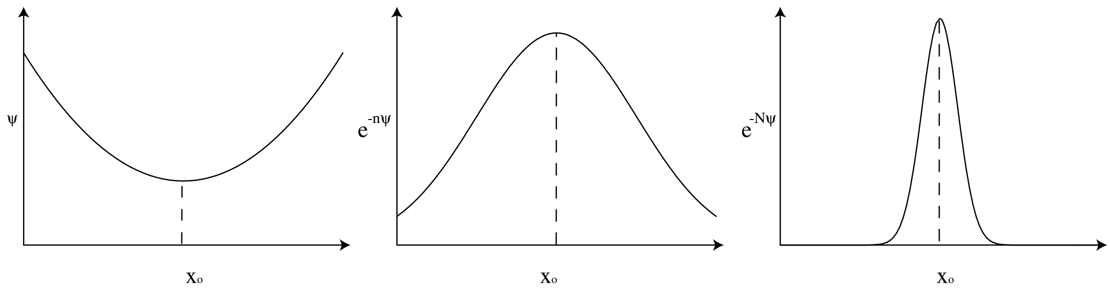

Figure 2: The contribution near the minimum at x_0 is more dominant for $N >> n$

For $\psi(x)$ having an interior minimum at x_0, a natural thing to try is to integrate from $x_0 - \epsilon$ to $x_0 + \epsilon$, since as n gets large, the contribution to the integral will become more and more important in this region(see figure 2). By Watson's lemma, as n goes to infinity, the corrections to $I(n)$ are asymptotically small, thus justifying our dropping of the tails. So in this limit we may write:

$$I(n) \sim \int_{x_0-\epsilon}^{x_0+\epsilon} e^{-n\psi(x)} dx \tag{2}$$

We also choose ϵ small enough so that we are able to write the Taylor expansion for $\psi(x)$ about x_0. Assuming that $\psi''(x_0) \neq 0$ write $\psi(x)$ as:

$$\psi(x) = \psi(x_0) + \frac{1}{2}\psi''(x_0)(x - x_0)^2 + \mathcal{O}\left((x - x_0)^3\right) \tag{3}$$

Note that in the case where $\psi''(x_0)$ is zero, we expand $\psi(x)$ further to include the next nonzero term in (3). Here we are dealing with ϵ being dependent on n so that what we are getting in (2) is an accurate asymptotic estimate of $I(n)$. Substitution of the expansion in equation (2) yields:

$$I(n) \sim e^{-n\psi(x_0)} \int_{x_0-\epsilon}^{x_0+\epsilon} e^{-\frac{n}{2}\psi''(x_0)(x-x_0)^2} dx \tag{4}$$

Even though we derived (4) by cutting up the tails of the integral, we now let $\epsilon \to \infty$. Putting the tails back introduces asymptotically small errors and doing so facilitates our evaluation of the integral. We finally obtain:

$$I(n) \sim e^{-n\psi(x_0)} \int_{-\infty}^{+\infty} e^{-\frac{n}{2}\psi''(x_0)(x-x_0)^2} dx$$

$$\Longrightarrow \boxed{I(n) \sim e^{-n\psi(x_0)} \sqrt{\frac{2\pi}{n\psi''(x_0)}} \text{ as } n \to \infty} \tag{5}$$

We now introduce a further complication in the integral of (1). In particular, as we have seen so far, we are mostly concerned with integrals where $\psi(x)$ is a complex function (such as the Fourier integrals). This type of integrals are asymptotically estimated by the method of steepest descents as we shall see in the next section.

2 Method Of Steepest Descents

The method of steepest descents is basically an extension of the Laplace method to integrals in the complex plane. We now have $\psi(z)$ be analytic and look at integrals of the form:

$$I(n) = \int_{\mathcal{C}} e^{-n\psi(z)} dz \tag{6}$$

where $\mathcal{C}$ is a contour in the complex plane. Note that $\psi(z)$ does not have to be analytic everywhere as long as we stay within a region where $\psi(z)$ is analytic along the contour. Having in mind what we did in the previous section, we attempt to apply the same ideas in the complex plane. In this case however we must deform the contour of integration in such a way that the major contribution to the integrand comes from a small portion of the new path of integration, this contribution becoming more and more dominant as n grows.

The size of the integrand is determined by its modulus and the key thing here is to notice that:

$$\left|e^{-n\psi(z)}\right| = \left|e^{-\psi(z)}\right|^n = \left|e^{-\Re e\{\psi(z)\}}\right|^n \left|e^{-i\Im m\{\psi(z)\}}\right|^n = \left|e^{-n\Re e\{\psi(z)\}}\right| \tag{7}$$

i.e. the imaginary part of ψ does not contribute to it. Hence the minimum of $\Re e\,\{\psi(z)\}$ will dominate the integral in the same way as the Laplace Integral. However, recalling the maximum modulus principle from complex analysis we find that for $e^{-\psi}$ analytic, $\left|e^{-\psi}\right|$ has no zeros and cannot attain a maximum in the interior of any finite domain, except possibly along its boundaries. Nevertheless, at points for which $\psi'(z) = 0$, there can exist saddle points in the modulus, i.e. points which can be maxima along certain contours and minima along others. Hence it is usually desirable to deform our contour so that it passes trough the saddle point, but this should be done so that the integrand contribution is maximized in the neighborhood of the saddle point and decays exponentially away from it.

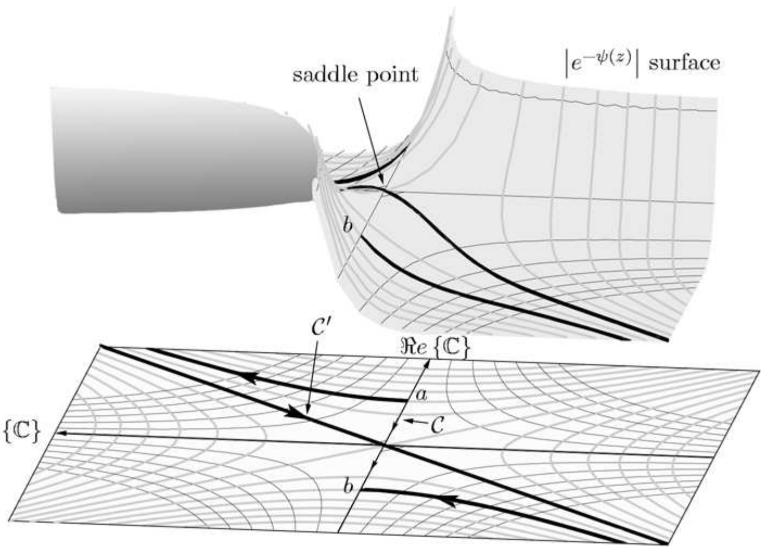

Figure 3: Example for the case when $\psi(z) = -iz^2$ and when $\mathcal{C}$ is along the real axis. The contours of constant $\Im m\,\{\psi(z)\}$ and $\Re e\,\{\psi(z)\}$ correspond to the grey and black lines respectively. These contours are projected onto the $\left|e^{-\psi(z)}\right|$ surface

Consider now the path of steepest accent towards the saddle point and the steepest descent away from it. Such a path is Parallel to $\nabla \Re e \{\psi(z)\}$. By a Cauchy Riemann condition, this path is normal to $\nabla \Im m \{\psi(z)\}$ (except ofcourse at the saddle points) and hence $\Im m \{\psi(z)\}$ remains constant along the path. Therefore, by deforming our initial contour $\mathcal{C}$ to a new contour $\mathcal{C}'$ along which $\Im m \{\psi(z)\}$ is constant, we reduce our original integral to a Laplace integral:

$$I(n) = \int_{\mathcal{C}} e^{-n\psi(z)} dz = e^{-ni\Im m\{\psi(z)\}} \int_{\mathcal{C}'} e^{-n\Re e\{\psi(z)\}} dz \tag{8}$$

We now calculate the leading order contribution to the integral from a saddle point at z_0.Note that in case we have multiple saddle points care should be taken when selecting which one(s) should be included in our contour. For the problem at hand assume that the saddle we are dealing with is of second order ($\psi''(z_0) \neq 0$) and expand near z_0 :

$$\psi(z) = \psi(z_0) + \frac{1}{2}\psi''(z_0)(z - z_0)^2 + \mathcal{O}\left((z - z_0)^3\right) \tag{9}$$

Using this expansion, our original integral, in the limit of large n tends asymptotically to:

$$I(n) \sim e^{-n\psi(z_0)} \int_{\mathcal{C}'} e^{-\frac{n}{2}\psi''(z_0)(z-z_0)^2} dz \tag{10}$$

Proceeding as before one can obtain a similar relation as in equation (5). This concept of contour deformation to a steepest descents path is illustrated in figure 3, when $\psi(z) = -iz^2$ and $\mathcal{C}$ is along the real axis. Next lecture we conclude our discussion on the saddle point method by applying it to derive a uniformly valid asymptotic approximation for the Bernoulli random walk.