Lecture 6: Fat Tails

R. Haghgooie

Department of Chemical Engineering, MIT

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Today we are discussing Divergent Moments, or 'Fat Tails'

$$\text{Assume } | m_1 |, | m_2 | < \infty$$

$$\text{What if } | m_l | \to \infty, \; l > 2 \text{ ?}$$

(CLT still holds but Berry Esseen Theorem and Edgeworth expansion doesn't apply)

1 Example

Here we will do an example. Consider the following PDF with iid steps.

$$p(x) = \frac{A}{1+x^4} \to \langle | x |^{\mu} \rangle = \infty \text{ for } \mu \geq 3 \tag{1}$$

The Fourier Transform for $p(x)$ is:

$$\hat{p}(k) = A \int_{-\infty}^{\infty} \frac{e^{ikx}}{1+x^4} dx \tag{2}$$

Another way to write this transform is

$$\begin{aligned} \lim_{R\to\infty} \hat{p}(k) &= A \int_{-R}^{R} \frac{e^{ikz}}{1+z^4} dz \text{ (integrated over the real axis)} \\ &= 2A\pi \text{Res}(e^{i\pi/4}, e^{3i\pi/4}) \end{aligned} \tag{3}$$

For an example of this integral see Lecture 9, Example 3 of the 18.325 notes. The transform then becomes

$$\begin{gathered} \hat{p}(k) = e^{-|k|/\sqrt{2}}(cos(k/\sqrt{2}) + sin(| k | /\sqrt{2})) \\ \text{where } A = \frac{\sqrt{2}}{\pi} \text{ by setting } \hat{p}(0) = 1 \end{gathered} \tag{4}$$

If we allow $k \to 0$ then the expansion of $\hat{p}(k)$ is

$$\hat{p}(k) \sim 1 - \frac{k^2}{2} + \frac{| k |^3}{3\sqrt{2}} - \frac{1}{24}k^4 + O(| k |^5) \tag{5}$$

where the analytic part, $1 - \frac{k^2}{2}$, gives $m_2 = 1$, while the singular part, $\frac{|k|^3}{3\sqrt{2}} - \frac{1}{24}k^4 + O(|\ k\ |^5)$, gives $m_3 = \langle |\ x\ |^3 \rangle \to \infty$

Taking the log of $\hat{p}(k)$ gives:

$$log(\hat{p}(k)) \sim \frac{-k^2}{2} + \frac{|\ k\ |^3}{3\sqrt{2}} - \frac{1}{6}k^4 + ..., k \to 0 \tag{6}$$

and thus:

$$P_N(x) = \int_{-\infty}^{\infty} e^{-ikx}\hat{p}(k)^N \frac{dk}{2\pi} \approx \int_{-\infty}^{\infty} e^{-ikx} e^{(\frac{-k^2}{2}+a|k|^3)N} \frac{dk}{2\pi} \quad \text{as } N \to \infty \tag{7}$$

Now we rescale with the following scaling:

$$w^2 = k^2 N,\ z = \frac{x}{\sqrt{N}}$$

giving:

$$\Phi_N(z) \approx \int_{-\infty}^{\infty} e^{-iwz} e^{\frac{-w^2}{2} + \frac{a|w|^3}{\sqrt{N}}} \frac{dw}{2\pi} \tag{8}$$

$$\Phi_N(z) = \sqrt{N} P_N(z\sqrt{N}) \tag{9}$$

If we keep z fixed while we allow $N \to \infty$ then we are inside the central region and equation (8) becomes:

$$\Phi_N(z) \approx \int_{-\infty}^{\infty} e^{-iwz} e^{-\frac{w^2}{2}} \left(1 + \frac{a\ |\ w\ |^3}{\sqrt{N}} + ...\right) \frac{dw}{2\pi} = \phi_0(z) + \frac{a\phi_1(z)}{\sqrt{N}} + ... \tag{10}$$

and ...

$$\begin{aligned} \phi_0(z) &= \frac{e^{-\frac{z^2}{2}}}{\sqrt{2\pi}} \\ \phi_1(z) &= \int_{-\infty}^{\infty} e^{-iwz} e^{-\frac{w^2}{2}}\ |\ w\ |^3\ \frac{dw}{2\pi} \end{aligned} \tag{11}$$

Since we know that $|\ w\ |^3 = w^3 sgn(w)$ we can write $\phi_1(z)$ as:

$$\phi_1(z) = 2\int_0^{\infty} e^{-\frac{w^2}{2}} w^3 cos(wz) \frac{dw}{2\pi} = \left(i\frac{\partial}{\partial z}\right) I(z) \tag{12}$$

where

$$\begin{aligned} I(z) &= -2i \int_0^{\infty} sin(zw) e^{-\frac{w^2}{2}} \frac{dw}{2\pi} \\ I(z) &= 2i\mathrm{Im}\left[\int_0^{\infty} e^{-izw-\frac{w^2}{2}} \frac{dw}{2\pi}\right] \equiv 2i\mathrm{Im}[J(z)] \end{aligned} \tag{13}$$

By completing the square for J(z) we find

$$J(z) = \frac{e^{-\frac{z^2}{2}}}{2\pi} \int_0^{\infty} e^{-(w+iz)^2/2} dw \equiv \frac{e^{-\frac{z^2}{2}}}{2\pi} L(z) \tag{14}$$

To solve for $L(z)$ we use Cauchy's Theorem which gives:

$$\begin{aligned} \oint_c e^{-\frac{z^2}{2}} dz &= 0 \\ \Rightarrow L(z) &= \sqrt{\frac{\pi}{2}} - i \int_0^z e^{t^2/2} dt \end{aligned} \tag{15}$$

Thus:

$$\begin{aligned} \mathrm{Im}[L(z)] &= - \int_0^z e^{t^2/2} dt \\ \mathrm{Im}[J(z)] &= -\frac{1}{2\pi} e^{-z^2/2} \int_0^z e^{t^2/2} dt \\ I(z) &= -\frac{i}{\pi} e^{-z^2/2} \int_0^z e^{t^2/2} dt \end{aligned} \tag{16}$$

and therefore

$$\phi_1 = -\frac{1}{2\pi} D''' \left(\frac{z}{\sqrt{2}} \right) \tag{17}$$

where $D(x) = e^{-x^2} \int_0^x e^{t^2/2} dt$ is Dawson's Integral. Differentiating Dawson's Integral gives

$$\begin{aligned} D'(x) &= 1 - 2xD(x) \\ D''(x) &= -2[x + (1 - 2x^2)D(x)] \\ D'''(x) &= 4(x^2 - 1) - 4x(2x^2 - 3)D(x) \end{aligned} \tag{18}$$

As $x \to \infty$ Dawson's Integral becomes

$$\begin{aligned} D(x) &\sim \frac{1}{2x} \left(1 + \frac{1}{2x^2} + \frac{3}{4x^4} + \sum_{m=3}^{\infty} \frac{(2m-1)!!}{2^m} x^{2m} \right) \\ D'''(x) &\sim \frac{-3}{x^4} \end{aligned} \tag{19}$$

This result finally gives

$$\phi_1(z) \sim \frac{6}{\pi z^4} \quad \text{as } z \to \infty \tag{20}$$

2 Edge of the Central Region

Where is the edge of the central region? It is found when $\phi_0(z) \sim \phi_1(z) \frac{a}{\sqrt{N}}$ as $z \to \infty$ and $N \to \infty$. This gives

$$\begin{aligned} \frac{e^{-z_{edge}^2/2}}{\sqrt{2\pi}} &\sim \frac{6a}{\pi z_{edge}^4 \sqrt{N}} \\ z_{edge} &\sim \sqrt{log(N)} + ... \\ x_{edge} &\sim \sqrt{Nlog(N)} \gg \sqrt{N} \quad \text{for } N \to \infty \end{aligned} \tag{21}$$

For our initial PDF,

$$\begin{aligned} p(x) &= \frac{\sqrt{2}/\pi}{1+x^4} \sim \frac{\sqrt{2}}{x^4\pi} \quad \text{as } x \to \infty \\ &= \frac{A}{x^4}; \quad A = \frac{\sqrt{2}}{\pi} \end{aligned} \tag{22}$$

if we claim that $\phi_1(z)$ is the leading term in the tails, then

$$P_N(x) \sim \frac{NA}{x^4} \tag{23}$$

which shows additivity in the tails