Lecture 5: The Continuum Limit

Ryan J. Larsen

Division of Engineering and Applied Sciences, Harvard University

February 20, 2003

In this lecture, we justify modifying the *Kramers-Moyall* expansion by relacing the moments in the expansion terms with cumulants. We do this by showing that the Green's function of the modified *Kramers-Moyall* equation exactly reproduces the original Bachlier equation. We also use scaling arguments to draw conclusions about the relative importance of the terms, and to justify neglecting higher-order terms in the expansion. The usefulness of the continuum approximation in solving discrete problems is briefly discussed.

1 The *Kramers-Moyall* expansion

When we are interested in the long time behavior of the random walker, and when the length scale of the random walker's individual steps, ΔX_n, is much shorter than the length scale, L, of P_N, it is possible to estimate P_N by taking the continuum limit of Bachelier's Equation. (As before, P_N is the probably distribution fuction of $\sum_{n=1}^{N} \Delta X_n$).

In Lecture 1 we showed that when $\Delta X_n / L \ll 1$, we may approximate P_N with a Taylor exansion in the following manner

$$P_{N+1} - P_N = \sum_{l=1}^{\infty} \frac{m_l}{l!} \left(-\frac{\partial}{\partial x} \right)^l P_N(x) \tag{1}$$

where $m_l =< y^l >= \int_{\infty}^{\infty} y^l p(y)\, dy$.

In lecture 4 we took the continuum limit by dividing both sides by the time interval, τ, between steps to obtain

$$\frac{P_{N+1} - P_N}{\tau} = \sum_{l=1}^{\infty} D_l \left(-\frac{\partial}{\partial x} \right)^l P_N(x) \tag{2}$$

where $D_l = m_l(\tau) / l!\tau$.

Then we interoduced the the function $\rho(x,t)$ as the continuous approximation of P_N. The continuous variable t represents the long time behavior of P_N such that $t = N\tau$, where N is, as usual, the number of steps taken. We required that $\rho(x, N\tau) = P_N(x)$.

As $\tau \to 0$ we obtained

$$\frac{\partial \rho}{\partial t} + \sum_{l=2}^{\infty} \frac{\tau^{l-1}}{l!} \left(-\frac{\partial}{\partial t}\right)^{l} \rho = \sum_{l=1}^{\infty} D_l \left(-\frac{\partial}{\partial x}\right)^{l} \rho. \tag{3}$$

The higher-order derivative terms are necessary in equation 3 due to the smoothness of $\rho(x,t)$. This equation without the second term on the left-hand side is known as the *Kramers-Moyall* expansion.

2 The Modified *Kramers-Moyall* expansion

In lecture 4, we claimed that the higher-order time derivatives on the left hand side of equation 3 can be removed by replacing the moments on the right-hand side with cumulants. Thus, equation 3 became

$$\frac{\partial \rho}{\partial t} = \sum_{l=1}^{\infty} \bar{D}_l \left(-\frac{\partial}{\partial x}\right)^{l} \rho \tag{4}$$

where $\bar{D}_l = c_l(\tau)/l!\tau$.

In lecture 4 we used an iterative process to demonstrate this explicitly for the first three terms of the series. We now demonstrate that this must be true for all terms by showing that the Green function of equation 4 reproduces exactly Bachelier's equation. The Green function is the solution to the PDE given the initial condition $\rho(x,0) = \delta(x)$.

We find the Green function of Equation 4 by first taking the Fourier transform

$$\frac{\partial \hat{\rho}}{\partial t} = \sum_{l=1}^{\infty} \bar{D}_l (ik)^{l} \hat{\rho}(k,t). \tag{5}$$

We obtained the Fourier transform of terms containing $\partial^l \rho/\partial x^l$ using the relation

$$\frac{\partial^l \rho}{\partial x^l} = \int_{-\infty}^{\infty} (-ik)^l \hat{p} \exp -ikx \frac{dk}{2\pi}. \tag{6}$$

The solution of equation 5 is

$$\hat{\rho}(k,t) = \hat{\rho}(k,0) \exp \sum_{l=1}^{\infty} \bar{D}_l (ik)^l t. \tag{7}$$

Thus, we obtain

$$\hat{\rho}(k,t) = \exp \sum_{l=1}^{\infty} N \frac{c_l}{l!} (ik)^l = \hat{\rho}(k)^N = \hat{P}_N(k). \tag{8}$$

3 Scaling the modified *Kramers-Moyall* expansion

We rewrite equation 4 to read

$$\frac{\partial \rho}{\partial t} + \frac{c_1}{\tau}\frac{\partial \rho}{\partial x} = \frac{c_2}{2\tau}\frac{\partial^2 \rho}{\partial \rho^2} - \cdots \tag{9}$$

which looks like the the familiar advection/diffusion equation. The coefficient of the second term on the left hand side has units of velocity v, and the coefficient of the first term on the right hand side has units of a diffusion constant D, which motivates us to write

$$c_1 \sim v\tau \tag{10}$$

$$c_2 \sim 2D\tau \tag{11}$$

$$\sigma \sim \sqrt{2D\tau}. \tag{12}$$

Thus, for short times τ the diffusion term dominates, and at longer times advection is more important. A natural choice for scaling the higher order terms in equation is to normalize the cumulants by the standard deviation, as in lecture 3. In that lecture we introduced normalized cumulants λ_n such that $\lambda_n = c_n/\sigma^n$. We thus obtain that for $n > 2$

$$D_n \sim \tau^{\left(\frac{n}{2}-1\right)} \tag{13}$$

causing us to conclued that higher-order terms may be neglected for small times.

Note how in the limit of $\tau \to 0$, c_2 may be replaced with m_2 without introducing significant errors. This is due to the fact that $c_2 = m_2 - {m_1}^2$, where $m_2 \sim \tau$ and ${m_1}^2 \sim \tau^2$.

We may arrive at the same scalings by non-dimensionalizing the modified *Kramers-Moyall* equation. We begin by introducing non-dimensional variables (denoted by tildas)

$$\tilde{t} = \frac{t}{T} \tag{14}$$

$$\tilde{x} = \frac{x - D_1 t}{L} \tag{15}$$

$$\tilde{\rho} = \frac{\rho}{\rho_0} \tag{16}$$

where we have yet to determine the typical length scale L.

This choice of variables allows us to rewrite the modified *Kramers-Moyall* expansion as

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} = \sum_{l=2}^{\infty}\left(\frac{\sigma^2 N}{L^2}\right)\frac{1}{2}\frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2} - \left(\frac{\sigma^3 N}{L^3}\right)\frac{\lambda_3}{3!}\frac{\partial^3 \tilde{\rho}}{\partial \tilde{x}^3} + \cdots \tag{17}$$

where we have used the relation $N = T/\tau$.

Notice how our choice of normalization of the x variable allowed us to remove the first term of the expansion involving D_1. This corresponds to studying the problem in a frame that moves with the random walkers, attached to mean of the distribution.

In order to choose the proper scaling of length L we use the "dominate balacne" method by assuming that two terms will have the same order of magnitude. Since the time derivative is clearly important at all times, we retain that term. The only logical choice to balance the time derivative is the diffusion term.

$$\frac{\sigma^2 N}{L^2} \sim 1 \tag{18}$$

which leads us to the familiar conclusion that $L = \sigma\sqrt{N}$. Had we tried to balance the next term with the time derivative, the diffusion term would have gone to infinity.

With this choice of L we rewrite the equation as

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} = D\frac{\partial^2 \tilde{\rho}}{\partial \tilde{x}^2} + \mathrm{O}\left(\frac{1}{\sqrt{N}}\right) \tag{19}$$

4 The Usefulness of the Continuum Limit

A common theme of this class will be that discrete problems are often easier to solve analytically, at least approximately in certain limits, by first taking the continuum limit. This is especially true when the coefficients D_1 or D_2 depend on time and/or space, or when the dimensionality of the problem is greater than 1 and complicated geometries occur. The continuum approximation may also simplify problems that contain nonlinearities, such as interactions between random walkers.