Lecture 25: Stochastic Laplacian (Diffusion-Limited) Growth

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1 Introduction

The Diffusion-Limited Aggregation (DLA) was first introduced by Witten and Sander [1] as a putative model for a process that leads to formation of "wispy" solid particles having essentially random and self-similar structure. The model is basically very simple: you put a "sticky" seed at the origin and start launching random-walking particles from infinity. Once such a particle hits the seed, it sticks to it thereby contributing to the growth of the aggregate. The process is highly stochastic and was shown to produce structures with power-law density correlation function and a certain characteristic appearance.

A number of computational techniques have been developed since; here we describe two methods:

1. *Direct simulation based on first-passage concepts* ([2], chapter 6.3.2 + references therein). In studying the morphology properties of the aggregate rather than the actual growth dynamics, efficient tricks were invented that use the first-passage time formalism to accelerate the simulation without loss of technical rigour. Namely, suppose the aggregate at some step of the simulation is bounded by a circle of radius R. Let at some time t_0 the diffusing particle cross this circle. The time that elapses before it sticks to the aggregate can be sufficiently reduced if we replace *henceforth* the periods that the particle spends outside the circle by single events of random sampling of points on the circle with probability distribution for first passage probability between the two points on the circle.

2. *Iterated conformal maps.* The bulk of the present lecture is devoted to the recently developed technique [3, 4], which, though being less effective computationally than first one, provides analytical insight into the structure of the problem.

2 Iterated Conformal Maps Method

The methods relies on the previously formulated ideas of the applicability of conformal maps to free-boundary problems. Namely, we solve the problem for a simple geometry in some complex plane ω. Then we try to find a conformal map that transforms the simple geometry to the original one.

Hastings and Levitov [3] took this approach one step further by formulating an iterative procedure that corresponds to random Laplacian growth.

2.1 The Formalism

The growing cluster is modeled by a sequence of domains $\mathcal{D}_0 \subset \mathcal{D}_1 \subset \mathcal{D}_2 \subset ...$ corresponding to subsequent growth steps in time. The particles concentration $c(z)$ obeys diffusion equation, which in the quasi-stationary approximation of slow growth is written as

$$\nabla^2 c(z) = 0 \quad \text{with } c(z) = \begin{cases} 0\,, & z \in \partial\mathcal{D}_{n-1}\,; \\ \frac{1}{2\pi}\log z\,, & z \to \infty\,. \end{cases} \tag{1}$$

Zero boundary condition on the cluster $\mathcal{D}_{n-1}$ describes sticking of the n-th particle upon arrival with probability one. The points of the cluster boundary $\partial\mathcal{D}_{n-1}$ where subsequent additions are made are selected randomly with the probability given by the so-called harmonic measure

$$dP = |\nabla c|\;|dz|\,, \quad dz \subset \partial\mathcal{D}_{n-1}\,, \tag{2}$$

where dl is boundary element of the cluster $\mathcal{D}_{n-1}$. As the domain changes, $... \to \mathcal{D}_n \to \mathcal{D}_{n+1} \to ...$, the problem (1) has to be solved again for every new domain to determine from (2) the new particle position probability.

The domains $\mathcal{D}_n$ are now represented by a sequence of analytic functions g_n that are chosen so that each of them defines a conformal one-to-one mapping of the unit disk $|\omega| \leq 1$ on the domain $\mathcal{D}_n$, including boundary. Adding a new object to the cluster at the n-th growth step is described by changing the mapping g_n as follows:

$$g_n(\omega) = g_{n-1}(f_{\lambda_n,\theta_n}(\omega)), \quad F_0(\omega) = \omega. \tag{3}$$

Here the function $f_{\lambda_n,\theta_n}(\omega)$ maps the unit circle $|\omega| = 1$ onto a unit circle with a bump centered around the point $\omega = e^{i\theta_n}$ of the circle. The bump size is determined by the parameter λ_n as discussed below. The angle θ_n is chosen randomly at each growth step. The procedure is schematically represented in Fig. 1.

How does the probability measure transform under a conformal mapping? Since

$$P(z)|dz| = |\hat{n} \cdot \nabla c(z)|\;|dz| \tag{4}$$

and

$$\nabla_\omega = g'(\bar{\omega})\nabla_z \tag{5}$$

we have

$$P(z)|dz| = P(\omega)|d\omega|, \tag{6}$$

so that θ is distributed uniformly: $dP(\theta) = d\theta/2\pi$.

A natural question arises regarding the size and the shape of the bump on the circle.

1. *Bump size.* We naturally want to fix the size of the bump in the real space λ_0, so that the physical particles will have the same size. Thus, for the n-th iteration we have

$$\lambda_n = \frac{\lambda_0}{|g'_{n-1}(e^{i\theta_n})|} \tag{7}$$

2. *Bump shape.* The actual shape of the bump is circular with its center placed at the boundary of the circular region, at angle θ_n.

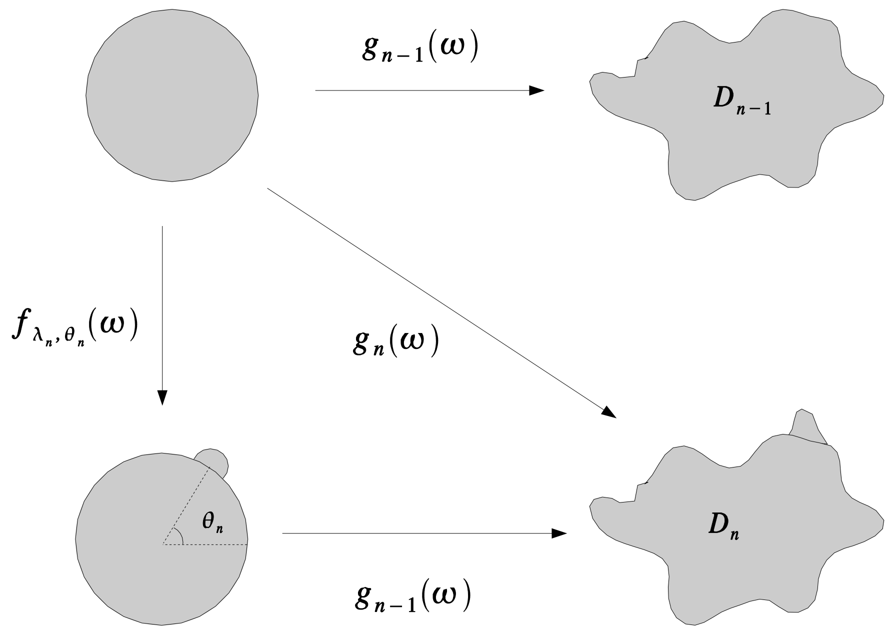

Figure 1: *The iterative procedure scheme.*

These statements become clear when we actually construct the map by the following procedure. First, we note that it is possible to map a unit circle onto a half plane by the map

$$z_1 = \frac{\omega - 1}{\omega + 1}. \tag{8}$$

Next, we employ the (scaled) Joukowski transform that maps a half-plane onto a half plane with a semi-circular bump of radius λ:

$$-iz_1 = \left(\frac{iz_2}{\lambda} + \frac{\lambda}{iz_2}\right)\frac{\lambda}{2} \tag{9}$$

so that

$$z_2 = z_1 + \sqrt{z_1^2 + \lambda^2}. \tag{10}$$

We also need to rescale this map by $1 + \sqrt{1+\lambda^2}$ in order that 1 will map to 1. This rescaling does introduce some inaccuracy in the circular bump size, of order $O(\lambda^2)$, negligible for small λ. Finally, we apply the reverse to (8) transform

$$z = \frac{1 + z_3}{1 - z_3}. \tag{11}$$

To complete the procedure, we note that in order to produce a bump at a given angle θ, we have to rotate the original circle clockwize by θ and the circle with the bump counterclockwize by the same angle. The whole procedure is summarized in Fig. 2. For the "bump function" $f_{\lambda,\theta}(\omega)$ we therefore have

$$f_{\lambda,\theta}(z) = e^{i\theta} g^{-1}\left(\tilde{f}_\lambda(g(e^{-i\theta}z))\right), \tag{12}$$

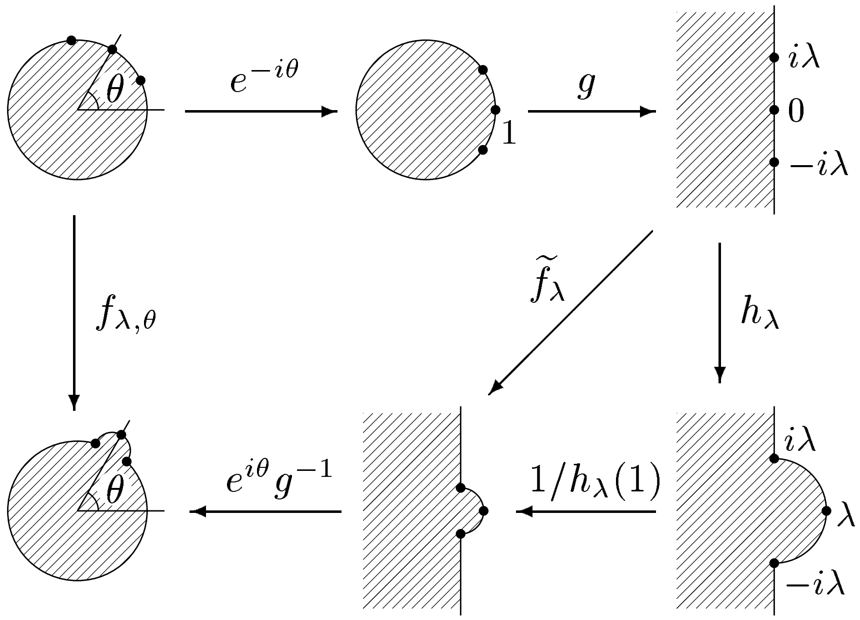

Figure 2: *The chain of conformal transformations that map a onto a circle with a bump (from [3]).*

2.2 Cluster Morphology

Figure 3 presents a beautiful DLA cluster produced by the above technique. The power of the approach, however, is that it is not merely a means of simulation; rather, it allows to make certain conclusions regarding the morphology of the cluster. For instance, it is possible to calculate the fractal dimension.

We note that $g_n(\omega)$ is a univalent map from the exterior of the unit circle. Therefore, it can be expanded in Laurent series including powers not higher than 1:

$$g_n(\omega) = \sum_{n=-\infty}^{1} A_m(n)\omega^m, \qquad |\omega| > 1. \tag{13}$$

As before, the area theorem is applicable

$$\mathcal{A}(n) = \pi \sum_{n=-\infty}^{1} m|A_m(n)|^2 = |A_1(n)|^2 - \pi \sum_{n=1}^{\infty} m|A_m(n)|^2. \tag{14}$$

Since the area must grow with time, the second (essentially negative) term can be discarded so that

$$\mathcal{A}(n) \sim |A_1(n)|^2. \tag{15}$$

Now, let $A_1(n) \sim n^{1/D_f}$. Then,

$$A_1(n) = \left.\frac{dg}{d\omega}\right|_{\omega\to\infty} = \prod_{m=0}^{n-1} f'_{\lambda_m,\theta_m}(\infty). \tag{16}$$

Here we used the identity

$$g' = \prod_{m=0}^{n-1} f'_{\lambda_m,\theta_m}, \tag{17}$$

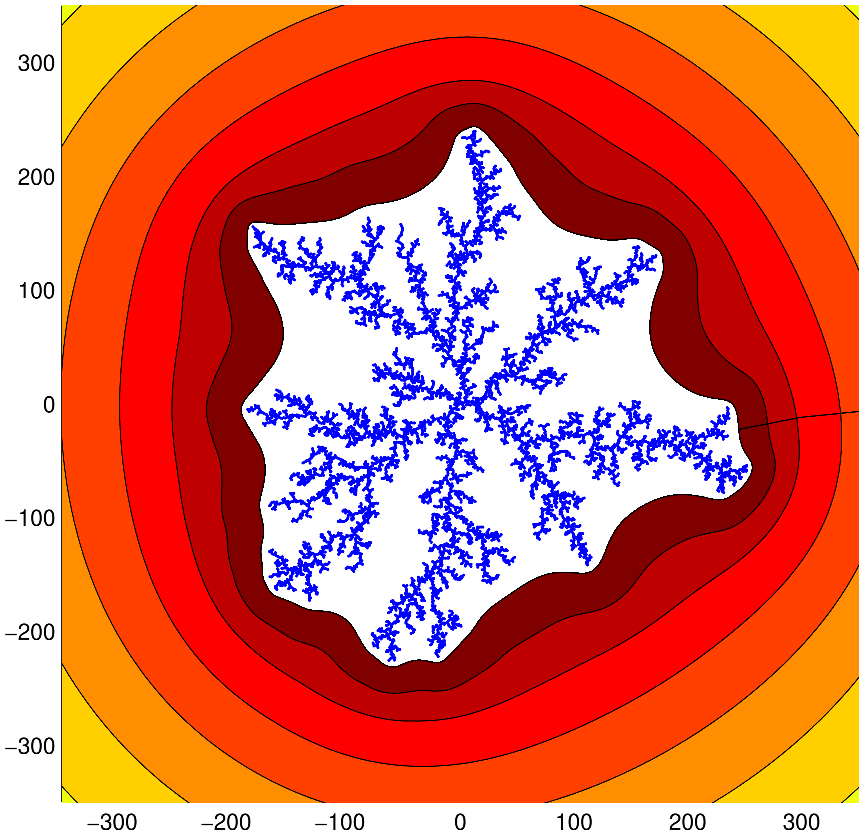

Figure 3: *Cluster obtained by iterative conformal mapping technique (courtesy of Mr. Jaehyuk Choi). The coloured regions correspond to the concentration profile.*

which can be obtained straightforwardly from (3) by using the chain rule. This identity also demonstrates the origin of the $O(n^2)$ complexity of the present algorithm: one should perform $O(n)$ calculations for the n-th particle.

The fractal dimension is now calculated as

$$D_f^{-1} = \lim_{n\to\infty} \frac{\log A_1(n)}{\log n} = \lim_{n\to\infty} \frac{1}{2\log n} \sum_{m=0}^{n-1} \log(1+\lambda_m^2). \tag{18}$$

Numerical simulations produce the value $D_f \simeq 1.71$.

We also can estimate the asymptotic scaling of λ_n with n. Since

$$\log A_1(n) \sim \frac{1}{2} \sum \lambda_m^2 \sim \frac{1}{D_f} \log n, \tag{19}$$

we obtain

$$\lambda_n^2 \sim \frac{2}{D_f n} \to 0. \tag{20}$$

Interestingly enough, the coefficients $A_1(n)$ appear to be log-normally distributed, with a variance that vanishes at large n. This counterintuitive finding was obtained numerically and there is stil no analytical explanation of this phenomenon.

References

[1] T. A. Witten and L. M. Sander, *Phys. Rev. Lett.* **47**, 1400 (1981).

[2] S. Redner, *A Guide to First-Passage Processes*, Cambridge University Press, New York, 2001.

[3] M. B. Hastings and L. S. Levitov, *Physica D* **116**, 244 (1998).

[4] M. G. Stepanov and L. S. Levitov, *Phys. Rev. E* **63**, 061102 (2001).