Lecture 24: Laplacian Growth II

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In the last lecture, we looked at the continuous dynamics of a freely moving domain. Using a mapping from the boundary in the z-plan to the unit circle in the ω-plan, $z = g(\omega, t)$, we have proved the following relation for the finite diffusive growth:

$$\mathrm{Re}\left(\overline{\omega g'}\, g_t\right) = 1 \quad \text{on } |\omega| = 1 \tag{1}$$

1 Laurent Expansion of $g(\omega, t)$

We know from the previous lecture that $g(\omega, t)$ must be a univalent (1-1, conformal) map from $\{|\omega| \geq 1\}$ to $\Omega_z(t)$. Thus, we write the Laurent Expansion:

$$g(\omega, t) = \sum_{n=-\infty}^{1} A_n(t)\omega^n \quad \text{for } |\omega| \geq 1$$

where the sum goes up to $n = 1$ to ensure that g is a 1-1 map. We calculate:

$$\begin{aligned} g'(\omega, t) &= \sum_{n=-\infty}^{1} nA_n(t)\omega^{n-1} \\ \omega g'(\omega, t) &= \sum_{n=-\infty}^{1} nA_n(t)\omega^n \\ \overline{\omega g'(\omega, t)} &= \sum_{n=-\infty}^{1} n\overline{A_n(t)}\omega^{-n} \end{aligned}$$

since $\overline{\omega} = \omega^{-1}$ for $|\omega| = 1$. Also:

$$g_t(\omega, t) = \sum_{n=-\infty}^{1} \dot{A}_n(t)\omega^n$$

to finally write from (1):

$$\mathrm{Re}\left(\sum\sum_{n,n'=-\infty}^{1} n\overline{A_n(t)}\dot{A}_{n'}(t)\omega^{n'-n}\right) = 1 \tag{2}$$

1.1 The Area Equation

Since the Laurent coefficients are unique, one must verify the following relation from (2) for the terms $n = n'$:

$$\mathrm{Re}\left(\sum_{n=-\infty}^{1} n\overline{A_n(t)}\dot{A}_n(t)\right) = 1$$
$$= \frac{1}{2}\sum_{n=-\infty}^{1} n\left(\overline{A_n(t)}\dot{A}_n(t) + A_n(t)\dot{\overline{A_n(t)}}\right)$$
$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\sum_{n=-\infty}^{1} n\left|A_n(t)\right|^2$$

that is:

$$\sum_{n=-\infty}^{1} n\left|A_n(t)\right|^2 = 2t + \sum_{n=-\infty}^{1} n\left|A_n(0)\right|^2 \tag{3}$$

Aside, we can notice that the area $\mathcal{A}(t)$ of the domain $\mathcal{C}_z(t)$ can be computed the following way:

$$\mathcal{A}(t) = \frac{1}{2i}\oint_{\mathcal{C}_z(t)} \overline{z}\mathrm{d}z = \frac{1}{2i}\oint_{|\omega|=1} \overline{g}g'\mathrm{d}\omega$$

by using the change of variable $z = g(\omega, t)$, with $\overline{z} = \overline{g}$ and $\mathrm{d}z = g'\mathrm{d}\omega$. Plugging the Laurent expansion of $g(\omega, t)$:

$$\frac{1}{2i}\oint_{|\omega|=1} \overline{g}g'\mathrm{d}\omega = \frac{1}{2i}\oint_{|\omega|=1}\left(\sum\sum_{n,n'=-\infty}^{1} n\overline{A_n(t)}A_{n'}(t)\overline{\omega}^n\omega^{n'-1}\right)\mathrm{d}\omega$$
$$= \frac{1}{2i}\sum\sum_{n,n'=-\infty}^{1} n\overline{A_n(t)}A_{n'}(t)\oint_{|\omega|=1}\frac{\mathrm{d}\omega}{\omega^{n-n'+1}}$$
$$= \pi\sum_{n=-\infty}^{1}\left|A_n(t)\right|^2$$

since:

$$\oint_{|\omega|=1}\frac{\mathrm{d}\omega}{\omega^{n-n'+1}} = \begin{cases} 2\pi i & \text{if } n = n' \\ 0 & \text{if } n \neq n' \end{cases}$$

Finally, (3) can be written:

$$\mathcal{A}(t) = \pi\sum_{n=-\infty}^{1} n\left|A_n(t)\right|^2 = \mathcal{A}(0) + 2\pi t \tag{4}$$

which finds its physical interpretation from te fact that the total flux from ∞ is constant and equal to 2π.

1.2 Other terms

We are now interested by the terms of (2) for which $n \neq n'$:

$$\sum_{n=-\infty}^{1-k} \left[n\dot{A}_{n+k}(t)\overline{A_n(t)} + (n+k)A_{n+k}(t)\overline{\dot{A}_n(t)} \right] = 0 \quad \text{for} \quad k = n' - n = 1, 2, \ldots \tag{5}$$

Using Laurent expansion, we simplified considerably the problem, since we went from an integral partial differential equation (1) to a system of first order non-linear ordinary differential equation.

2 Exact Solutions

2.1 Circle

For a simple growing circle:

$$g(\omega, t) = A_1(t)\omega$$

From (4), we get:

$$\pi \left| A_1(t) \right|^2 = \pi + 2\pi t$$

Since $\mathcal{A}(0) = \pi$ for the unit circle. Choosing $A_1(t)$ real:

$$A_1(t) = \sqrt{1 + 2t}$$

is simply the radius of the growing circle.

2.2 Ellipse

For the ellipse, we verify:

$$g(\omega, t) = A_1(t)\omega + \frac{A_{-1}(t)}{\omega}$$

Indeed, this mapping is the sum of 2 circular, counter rotating mapping with same rotation angle. Assuming $\left|A_1(t)\right| < \left|A_{-1}(t)\right|$, this produces an ellipse with semi-major $\left|A_1(t)\right| + \left|A_{-1}(t)\right|$ and semi-minor $\left|A_1(t)\right| - \left|A_{-1}(t)\right|$. Without loss of generality, we will write:

$$A_1(0) = 1$$
$$A_{-1}(0) = c$$

where c is real and $0 < c < 1$. It can be shown that $A_1(t)$ and $A_{-1}(t)$ remain real at all times. The area equation (4) becomes:

$$\left|A_1(t)\right|^2 - \left|A_{-1}(t)\right|^2 = 2t + 1 - c^2$$

and (5) gives:

$$-\dot{A}_{-1}(t)A_1(t) + A_1(t)\dot{A}_{-1}(t) = 0 \quad \Rightarrow \quad \frac{\dot{A}_1(t)}{A_1(t)} = \frac{\dot{A}_{-1}(t)}{A_{-1}(t)}$$

and after integration:

$$A_1(t) = \frac{A_{-1}(t)}{c}$$

Finally, we write:

$$g(\omega, t) = \sqrt{1 + \frac{2t}{1 - c^2}} \left(\omega + \frac{c}{\omega} \right)$$

Taking $c = 1$ leads to the circular mapping.

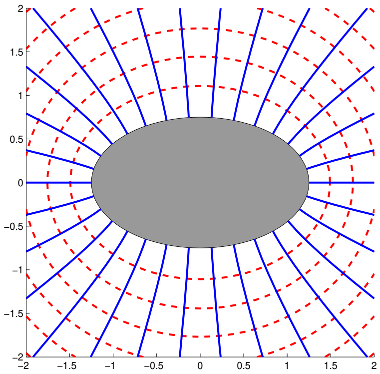

Figure 1: Initial mapping as given by $g(\omega, 0) = \omega + c/\omega$, with $c = 0.25$.

2.3 M-Fold Perturbation of the Circle

We consider now the following mapping:

$$g(\omega, t) = A_1(t)\omega + \frac{A_{1-M}(t)}{\omega^{M-1}}$$

Again, without loss of generality, we will write:

$$A_1(0) = 1$$
$$A_{1-M}(0) = \frac{c}{M-1}$$

Let us see what the following mapping gives:

$$g(\omega, 0) = \omega + \frac{c}{(M-1)\omega^{M-1}} \tag{6}$$

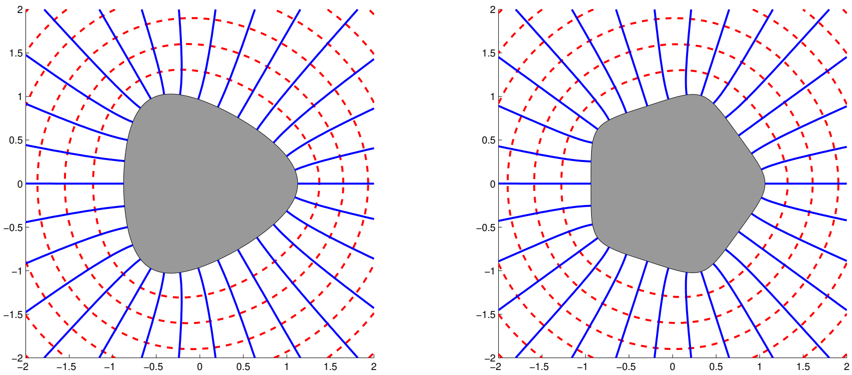

Figure 2: Mapping consisting of an M-fold perturbation of a circle, as given by (6): on the left, $M = 3$, on the right $M = 5$. For both, $c = 0.25$.

We evaluate:

$$g'(\omega, 0) = 1 - c\omega^{-M}$$

indicating that if $c \geq 1$, the mapping loses its conformality in the domain $|\omega| \geq 1$ at M points on a circle: $\omega_n = c^{1/M} e^{i2\pi n/M}$, $n = 0, 1, \ldots, M-1$. However, if $c < 1$, conformality is never lost.

The area equation (4) gives:

$$\left|A_1(t)\right|^2 - (M-1)\left|A_{1-M}(t)\right|^2 = 2t + 1 - \frac{c^2}{M-1}$$

and (5) gives:

$$-(M-1)\dot{A}_1(t)A_{1-M}(t) + A_1(t)\dot{A}_{1-M} = 0 \quad \Rightarrow \quad (M-1)\frac{\dot{A}_1(t)}{A_1(t)} = \frac{\dot{A}_{1-M}(t)}{A_{1-M}(t)}$$

which gives after integration:

$$A_1(t)^{M-1} = \frac{M-1}{c} A_{1-M}(t)$$

We finally get:

$$g(\omega, t) = A_1(t)\left[\omega + \frac{cA_1(t)^{M-2}}{(M-1)\omega^{M-1}}\right]$$

where $A_1(t)$ solves:

$$\left|A_1(t)\right|^2 - \frac{c^2}{M-1}\left|A_1(t)\right|^{2M-2} = 2t + 1 - \frac{c^2}{M-1}$$

Following the previous discussion, we want to check when the mapping loses its conformality on $|\omega| = 1$, that is we want to find t_c such that $cA_1(t_c)^{M-2} = 1$. Plugging this relation into the equation above gives:

$$t_c = \frac{1}{2}\left[\frac{c^2}{M-1} + \frac{M-2}{M-1}\left(\frac{1}{c}\right)^{\frac{2}{M-2}} - 1\right]$$

which has a positive finite value for any $M \geq 3$ and $0 < c < 1$. Thus, the mapping will lose its conformality after a finite time, as illustrated in figure 3.

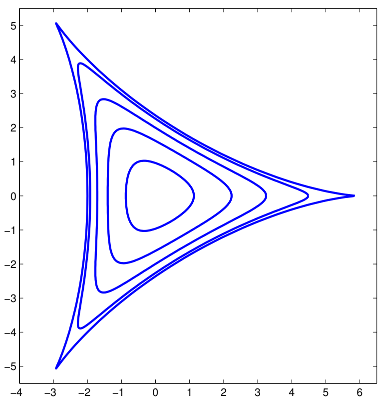

Figure 3: Evolution of a mapping consisting of a 3-fold perturbation of a circle ($M = 3$). In this case, $c = 0.25$ so that $t_c = 3.51563$. The contours are plotted at the times $t = 0, 1, 2, 3$ and 3.51563.

2.4 Additional Remark

For a mapping with a finite Laurent expansion:

$$g(\omega, t) = \sum_{n=-N}^{1} A_n(t)\omega^n \quad \text{with} \quad N < \infty$$

we can still show that (5) leads to:

$$A_1(t)^N = \mathrm{C}^{\mathrm{te}} A_{-N}(t) \tag{7}$$

On the other hand, the area rule (4) gives:

$$\left|A_1(t)\right|^2 - \sum_{n=1}^{N} n\left|A_{-n}(t)\right|^2 = 2t + \text{const}$$

showing that $A_1(t)$, which gives the only positive term in the left hand side term, must grow at least as $\sqrt{t}$. Then from (7), we conclude that $A_{-N}(t)$ grows at least as $t^{N/2}$ which is in contradiction with the area rule. In order to solve this, the mapping must reach a singularity in a finite time t_c, as in the previous section.