Laplacian Growth Phenomena, I.

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1 Continuous Laplacian (diffusive) growth phenomena

We will now look at a kind of problem where moving from a continuous to a discrete framework leads to a singular perturbation which depends on the nature of the discreteness of the system.

The basic problem is that we have a bounded region $\Omega_z(t)$ that has a boundary surface $\partial\Omega_z(t)$. Outside of the region $\Omega_z(t)$, we solve the Laplace's equation for quasi-steady normal diffusion,

$$\nabla^2\phi = 0, \tag{1}$$

where Equation (1) holds in the "quasi-steady approximation." That is, we are solving

$$\frac{\partial\phi}{\partial t} = D\nabla^2\phi,$$

in a regime in which $\partial\phi/\partial t$ can be neglected. The boundary $\partial\Omega_z(t)$ is assumed to be an equipotential, or constant-field, surface satisfying the boundary condition $\phi = 0$. In addition there is a "free boundary" moving at velocity $\mathbf{v}$, which is given by the normal of the gradient field:

$$\mathbf{v} = \alpha\left(\hat{\mathbf{n}}\cdot\nabla\phi\right)\hat{\mathbf{n}} = \alpha\nabla\phi.$$

Alternatively, we can write (1) as

$$\frac{\partial\phi}{\partial t} = -\nabla\cdot\mathbf{F},$$

where the flux $\mathbf{F}$ is defined as

$$\mathbf{F} = -D\nabla\phi.$$

The inherent challenge is that non-linearity arises in the problem because the location of the free boundary $\partial\Omega_z(t)$ must be computed before we can actually apply the boundary conditions!

Some physical examples of this process are:

Example 1 ***Solidification from an undercooled melt or melting of a superheated solid.***

Solidification or melting begins with a seed particle that, at a particular critical temperature T_c, begins to absorb heat from or release heat into the system. For example, in a solidification process, we have a solid seed in a liquid system with far-field temperature $T \to T_\infty \geq T_{melt}$, while the seed itself is at $T = T_0 < T_{melt}$. Eventually, we expect that the whole system will be converted from

liquid to solid. If the phase transition is "first-order," then it will generate latent heat which must be released to the environment. The heat flux

$$\mathbf{J} = -\kappa \nabla T,$$

where κ is the thermal diffusivity constant, leads to growth of the velocity of the boundary: $\mathbf{v} \propto \mathbf{J}$. The equation which describes the growth of the crystal is the heat equation

$$\frac{\partial T}{\partial t} = \mathcal{D} \nabla^2 T. \tag{2}$$

When $\partial T/\partial t$ does not vanish, (2) is known as the Stefan problem; this is the *classic free-boundary problem. Roughly, we can imagine that in the limit of slow solidification, the growth will be quasi-steady, and we can make a lot of progress in two dimensions.*

Example 2 ***Viscous fingering in a porous medium or Hele-Shaw cell.***

Darcy's law in a porous medium states that the velocity is given by $\mathbf{u} = \mu \nabla P$, where the velocity satisfies the continuity equation and the pressure satisfies Laplace's equation:

$$\nabla \cdot \mathbf{u} = 0,$$
$$\nabla^2 P = 0.$$

If we assume that we have a bubble of less viscous fluid at an essentially constant pressure $P = P_0$ pumped into a more viscous fluid moving in a field $\mathbf{u}$ proportional to ∇P, then the velocity of the boundary is given by

$$\mathbf{v} = \mathbf{u} = \mu \nabla P. \tag{3}$$

Saffman and Taylor (1958) studied a quasi-two-dimensional system in which a fluid is trapped between two flat plates, known as a Hele-Shaw cell. A second fluid is injected through a hole in the top of the system, which leads to a pressure gradient. The pressure gradient induces a velocity field; the height-averaged velocity is also given by (3), where μ is the viscosity of the fluid.

As we have indicated above, the free boundary problem (1) is relatively intractible in three dimensions; we can make analytical progress in two dimensions by using complex analysis.

2 A few concepts from complex analysis

1. A complex number z can be written in either Cartesian coordinates $z = (x, y) = x + iy$ or in Euler form $z = (r, \theta) = re^{i\theta}$, where r is the modulus and $\theta = \arg(z)$ is the argument.

2. Addition of complex numbers written in Cartesian coordinates proceeds just like vector addition: $z_1 + z_2 = (x_1 + x_2, y_1 + y_2)$.

3. Multiplication of complex numbers in polar notation proceeds by multiplying moduli and adding arguments:

   $$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

   Thus multiplication by z is equivalent to a stretching by $|z|$ followed by a rotation through $\arg(z)$.

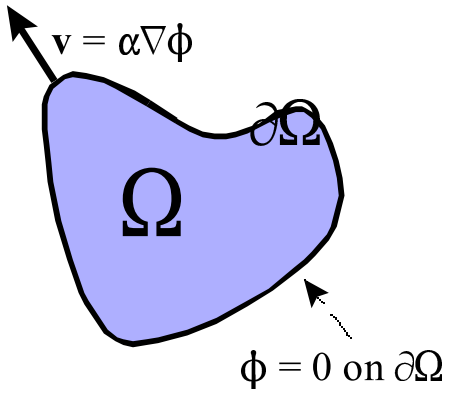

Figure 1: Region Ω on which Laplace's equation is solved for the problem of interface growth, with boundary conditions as shown.

4. If the derivative $f'(z)$ of a complex function exists in a set Ω, then $w = f(z)$ is said to be *analytic* on Ω. If f is analytic, then we can define the derivative as

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{4}$$

The central importance of (4) to complex analysis is that (4) must hold independent of the path taken from z to z_0. In addition, if $f(z) = u(z) + iv(z)$, and f is analytic, then we can write the complex (total) derivative as partial derivatives of the real and imaginary parts as

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y},$$

from which we can immediately write down the *Cauchy-Riemann conditions*:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \tag{5}$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{6}$$

5. A *conformal mapping* is a function $f(z)$ on a set Ω for which it analytic, and $f'(z) \neq 0$. For a conformal mapping, we can write

$$f(z) - f(z_0) \sim f'(z_0)(z - z_0) \tag{7}$$

as $z \to z_0$. Then, if we define $w = f(z)$ and $w_0 = f(z_0)$, we see that any small neighborhood of z_0 is stretched by $|f'(z_0)|$ and rotated by $\arg(f'(z_0))$. Thus, if we draw a grid around z_0, then the grid is only stretched and rotated *in the vicinity of* z_0. For points far from z_0, we need to include higher-order terms in (7), and therefore the grid will not be maintained after the mapping. However, the angle between two curves passing directly through z_0 must be preserved by a conformal mapping. [That is the meaning of the term "conformal."] Some sample mappings are shown in Figures 2 and 4.

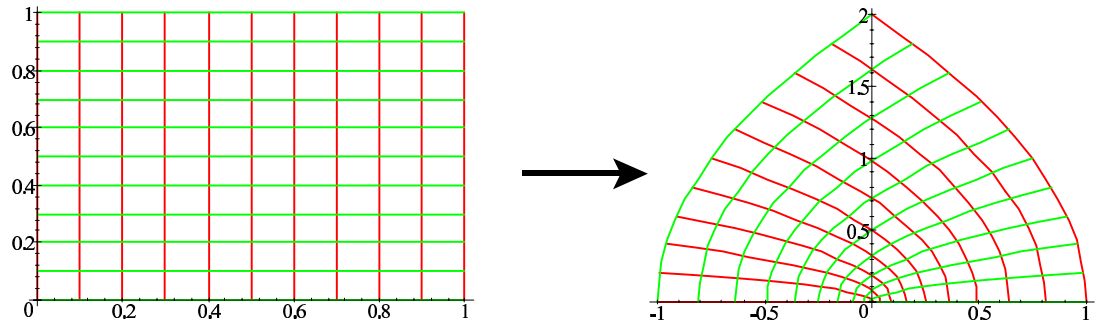

Figure 2: The conformal mapping $f(z) = z^2$. Note that the mapping is one-to-one except at the origin, where $f'(z) = 2z = 0$; the effect at the origin is to double the angle of z, leading to the spread of the mapping from the upper right quadrant to the upper half-plane.

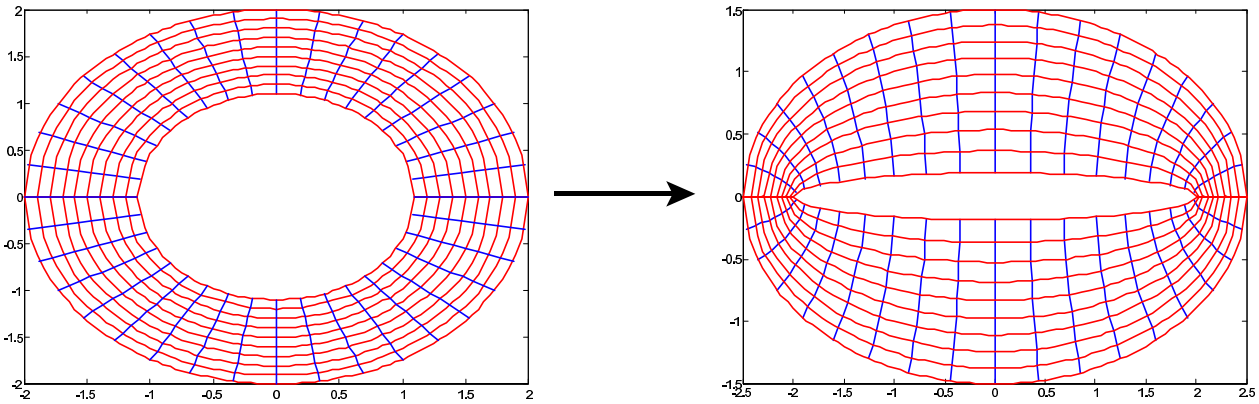

Figure 3: The conformal mapping $w = f(z) = z + z^{-1}$ for the annulus $1.1 \leq r \leq 2$. Angles between corresponding grid lines are preserved after the transformation. Note, though, that the unit circle has been excluded since $f'(z) = 1 + \log z = 0$ at for $z = 1$ and $z = -1$, and therefore the mapping is not conformal at those points.

6. The so-called *potential theory* of harmonic functions, which satisfy Laplace's equation, $\nabla^2\phi = 0$. Then the function ϕ can be defined as either the real or imaginary part of an analytic complex potential

$$\Phi = \phi + i\psi,$$

where the function ϕ is a potential function, while the harmonic conjugate ψ defines the streamfunction of the system. Consequently, one can think of solving Laplace's equation as the problem of determining an analytic function whose real part satisfies the appropriate boundary conditions. It should be noted that

$$\nabla\phi \equiv \frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial x} - i\frac{\partial\psi}{\partial x} = \overline{\frac{\partial\phi}{\partial x} + i\frac{\partial\psi}{\partial x}} = \overline{\Phi'(z)}, \tag{8}$$

where we have used the Cauchy-Riemann equations (5) and (6) repeatedly in (8).

3 Conformal mapping of potentials

Suppose that $\phi = \mathrm{Re}\,\Phi(w)$ in Ω_w such that $\nabla^2\phi = 0$. Then, if $w = f(z)$ is conformal in Ω_w, then

$$\phi = \mathrm{Re}\,\Phi(f(z))$$

is a solution to

$$\nabla^2\phi = 0$$

in $\Omega_z = f^{-1}(\Omega_w)$. Inserting the conformal mapping $f(z)$ into $\Phi(z)$ yields a function which is still analytic. For example, consider a rectangle Ω_w in the w-plane which is derived from a more complicated region Ω_z in the z-plane, then if $w = f(z)$ and $\phi = \mathrm{Re}\,\Phi(w)$, then $\phi = \mathrm{Re}\,\Phi(f(z))$.

Fluxes also transform under conformal mappings. Consider $\mathbf{F}_w = \nabla_w\phi = \overline{\frac{d\Phi(w)}{dw}}$ defined in the simpler w-plane. Then, in the more complicated z-plane, we have

$$\mathbf{F}_z = \nabla_z\Phi = \overline{\frac{d\Phi(f(z))}{dz}}; \tag{9}$$

using the chain rule in (9) gives

$$\mathbf{F}_z = \nabla_z\Phi = \overline{\frac{d\Phi(f(z))}{dw}} \cdot \overline{\frac{df(z)}{dz}} = \mathbf{F}_w \cdot \overline{f'},$$

so again we have the "stretch and rotate" concept seen above.[1]

4 Conformal-map dynamics for Laplacian growth

In the w-plane, let us consider solving Laplace's equation in the right half-plane with the boundary condition $\phi = 0$ at $\mathrm{Re}\,w = 0$. Consider the complex flux $\nabla\phi \sim 1$ as $\mathrm{Re}\,w \to \infty$. The solution of Laplace's equation in this region is

$$\phi = \mathrm{Re}\,\Phi,\ \Phi = w.$$

[1] In his book *Visual Complex Analysis*, T. Needham calls the "stretch and rotate procedure" an "amplitwist."

Now assume that we have a time-dependent mapping $z = g(w,t)$ and a corresponding inverse mapping $w = f(z,t)$. Then, in the w-plane, we have $\phi = \mathrm{Re}\, w$ and $\Phi = w$; in the much more complicated region in the z-plane, the solution of $\nabla_z^2 \phi = 0$ yields a complex potential $\Phi = f(z,t)$ for which $\phi(z,t) = \mathrm{Re}\,\Phi(z,t)$.

We will now generate an equation of motion for $g(w,t)$ by following a point z on the boundary $\partial\Omega_z$. The mapping $w = f(z,t)$ allows us to determine a (possibly moving) "Lagrangian marker" on the imaginary axis which corresponds to z on $\partial\Omega_z$. In complex notation, the velocity of the boundary is $v = z_t$, where

$$v = z_t = \frac{dg(w(t),t)}{dt} = g' \frac{dw}{dt} + \frac{dg}{dt}. \tag{10}$$

Then

$$\mathbf{v} = \alpha \hat{\mathbf{n}} \cdot \nabla \phi, \tag{11}$$

where for convenience we will take $\alpha = 1$. In addition to the "kinematic" conditions (10) and (11), we have the "physical" constraint $v_z = \nabla_z \phi$. However, we don't want to work in the z-plane if we can possibly work in the much simpler w-plane. Then

$$v_z = \nabla_z \phi = \overline{f'} \nabla_w \phi. \tag{12}$$

However, for the simple half-plane problem we are considering,

$$\nabla \phi = \overline{\Phi'} = 1,$$

and therefore (12) reduces to

$$v_z = \overline{f'} = \frac{1}{\overline{g'}}, \tag{13}$$

where the last relation in (13) is established by the relationships between inverse functions and their derivatives. Consequently, we have

$$w_t \left|g'\right|^2 + \overline{g'} g_t = 1, \tag{14}$$

Since w_t is imaginary, $\mathrm{Re}(w_t) = 0$, so taking the real part of (14), we have on the imaginary axis

$$\mathrm{Re}\, \overline{g'} g_t = 1 \tag{15}$$

when we take $\mathrm{Re}(w) = 0$. We can now clearly see from (15) that the equation of motion for the free boundary is nonlinear. [This approach appears to have been first developed by Polubarinova-Kochina (1945) and independently by Galin (1945).]

Our goal, therefore is to solve (15), by any means possible. In this particular case, we have already made significant progress because we are solving a one-dimensional non-linear problem which is relatively tractable numerically instead of a harrowing two-dimensional problem over a complex geometry.

There are several classes of solutions; today we will discuss the class of so-called "traveling-wave solutions." Let $z = g(w,t)$ with $\mathrm{Re}\, w = 0$ be given by

$$z = g(w,t) = ut + g_1(w),$$

Then $g_1(w)$ is a constant mapping which is shifted over by the term ut. If we consider infinite profiles, where the profile changes from $z = -\infty$ to $z = +\infty$, from which we can obtain the trivial solution

$$g(w,t) = ut + \frac{w}{u},$$
$$g' = \frac{1}{u},$$
$$g_t = u,$$

and we therefore have a "shifting wall" moving from $y = 0$ to ut.

A less trivial solution can be derived by noting that since $\operatorname{Re} w = 0$ by construction, the real part of the derivative of w^2 is also zero. Therefore, we can include an "arbitrary amount of w^2" to the system, so that

$$g(w,t) = ut + \frac{w}{u} + w^2,$$

and

$$g' = \frac{1}{u} + 2w.$$

We are then warping the boundary of the w-plane from a line to a parabola. [Studying solidification in 1947, Ivantsov solved the three-dimensional Stefan problem for a paraboloid of revolution.] If we perturb the problem a little bit, we obtain side-branches or dendrites growing off our principal fingers. We can also show that there are no other traveling-wave solutions where g is analytic on $\operatorname{Re} w = 0$ (no cusps). In this model, this simplification arises from the fact that we are considering a finite seed and we have simple boundaries.

Another class of solutions, the class of periodic solutions, solves the Saffman-Taylor fingering problem. We now break the w-plane into periodic strips of thickness 2π. In each strip, we obtain a finger that stretches out to infinity which satisfies the viscous fingering problem in the Hele-Shaw cell. The solution they obtained, for $\nabla\phi \approx 1$ as $w \to \infty$, is

$$z = g(w,t) = \frac{t}{\lambda} + w + 2(1-\lambda)\log\left(1 + e^{-w}\right). \tag{16}$$

Note that (16) does satisfy

$$\operatorname{Re} \overline{g'} g_t = 1$$

on $\operatorname{Re} w = 0$, $w = i\theta$, $-\pi < \theta < \pi$.

Let us consider the unit circle to be the $\tilde{w}$-plane, which is mapped to from the w plane by $\tilde{w} = e^w$. Lines of constant v in the w-plane become rays in the $\tilde{w}$-plane, and lines of constant u become circles. In terms of $\tilde{w}$,

$$\begin{aligned} z &= \tilde{g}(\tilde{w},t) \\ &= g(e^w,t) \\ &= \log e^{t/\lambda}\tilde{w}\left(1 + \frac{1}{\tilde{w}}\right)^{2(1-\lambda)}. \end{aligned} \tag{17}$$

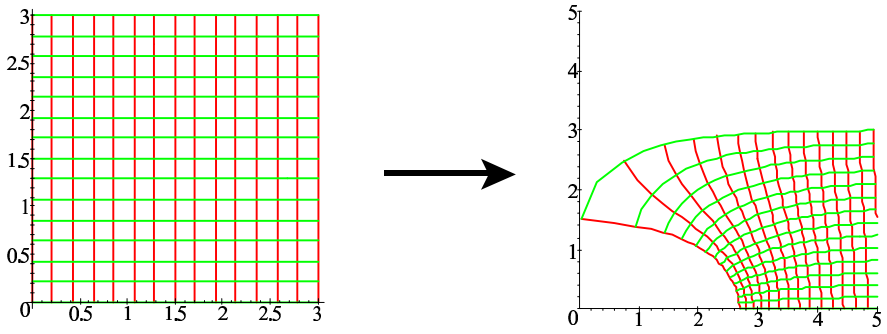

Figure 4: The conformal mapping (18) for $\lambda = 1/2$ and $t = 0$.

When $\lambda = 1/2$, (17) reduces to

$$z = \log e^{t/\lambda} (\tilde{w} + 1) . \tag{18}$$

In the $\tilde{w}$ plane, this corresponds to a series of growing circles such that it is tangent to the imaginary axis at the origin of the plane. The value λ determines the fraction of the channel that is filled by the finger, which extends from $-\lambda\pi i$ to $\lambda\pi i$. When $\lambda < 1/2$, the resulting shape is more wedge-like, leading to thinner fingers; when $1/2 < \lambda < 1$, we have a more limaçon shaped region, which leads to a thicker finger. However, λ can't be tuned, which means that some other physical effects, such as surface tension, must be present. Examples of the solutions for $\lambda = 1/4$, $\lambda = 1/2$, and $\lambda = 3/4$ are shown in Figures 5 through 7.

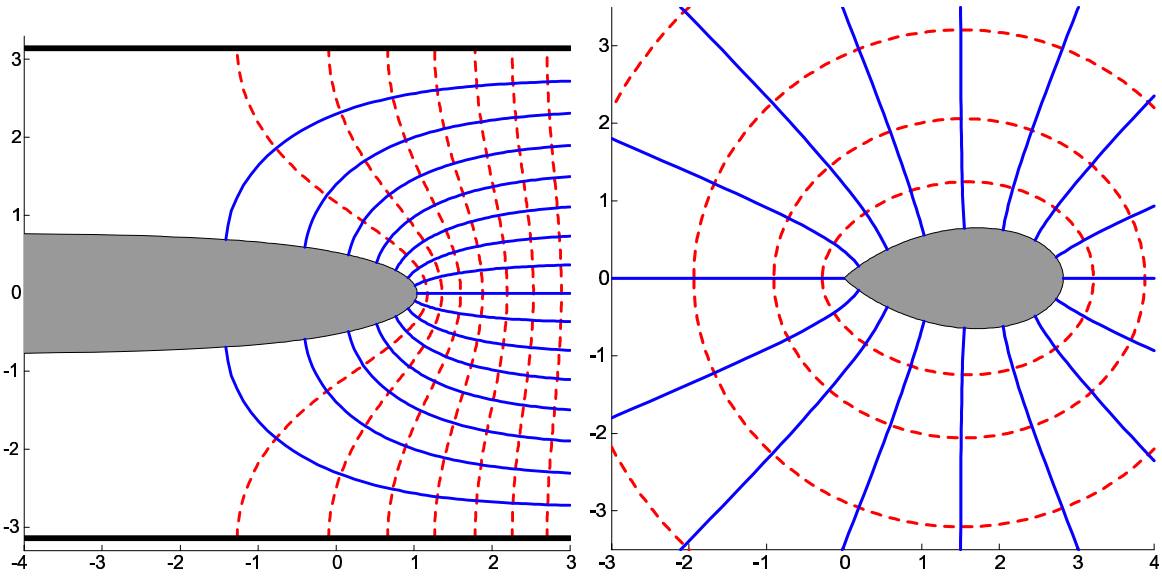

Figure 5: Solution of the Saffman-Taylor fingering problem with $\lambda = 1/4$ in the z-plane (left) and w-plane (right). Figures provided by J. Choi.

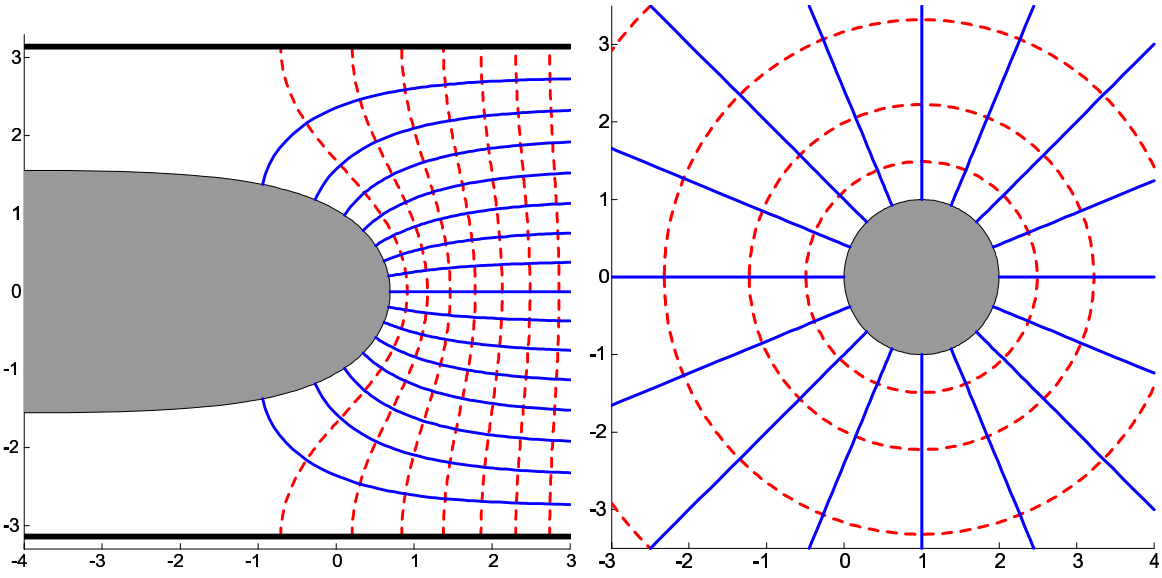

Figure 6: Solution of the Saffman-Taylor fingering problem with $\lambda = 1/2$ in the z-plane (left) and w-plane (right). Figures provided by J. Choi.

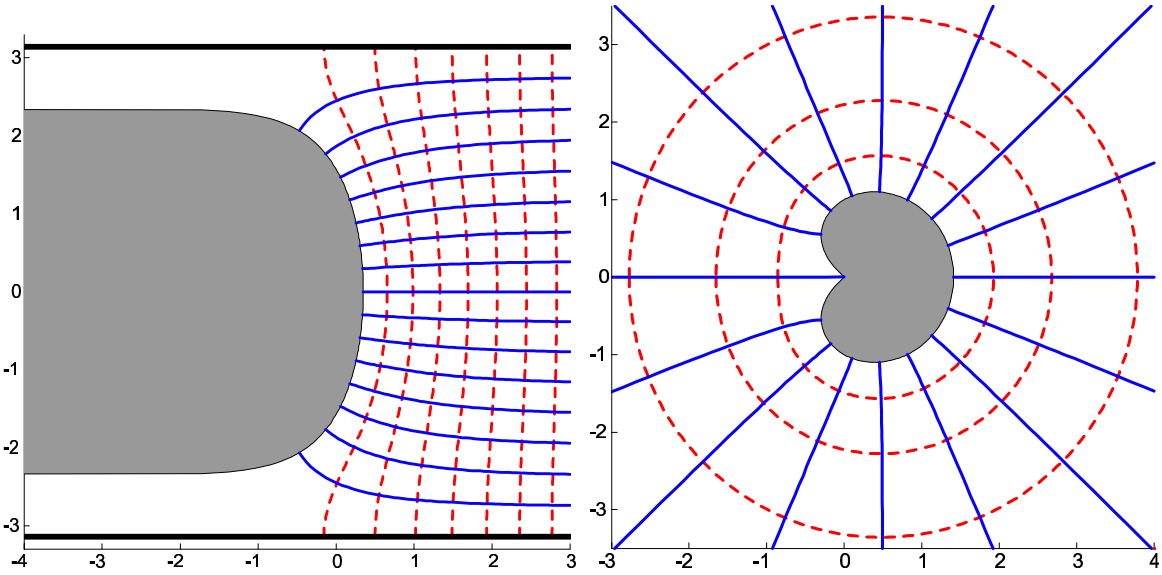

Figure 7: Solution of the Saffman-Taylor fingering problem with $\lambda = 3/4$ in the z-plane (left) and w-plane (right). Figures provided by J. Choi.