

Lecture 12: Levy Flights (=)

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1 Review

Recall (from Lectures 5 and 6) that if the first two moments of a PDF are finite and a higher moment diverges, then the Central Limit Theorem will still hold. However, that particular distribution will be characterized by “fat tails” with additive tail amplitude.

In this lecture we consider a general PDF with a diverging second moment, $\langle x^2 \rangle = \infty$, which can lead to anomalous diffusion.

2 Borderline cases of the CLT

First we consider *borderline cases of the CLT* to examine how the shape of the PDF and the diffusive scaling change in the large N limit. For iid random displacements:

$$\vec{X}_N = \sum_{n=1}^N \vec{x}_n$$

$p(x)$ PDF for x_n
 $P_N(x)$ PDF for N displacements

We define the Strong Central Limit Theorem as (M. Feller, *Introduction to Probability*, 1971):

Let $U(x) = \int_{-x}^x y^2 p(y) dy$

If $U(x) \sim L(x)$ where $L(x)$ is "slowly varying",
(i.e. $\lim_{x \rightarrow \infty} \frac{L(sx)}{L(x)} = 1$ for $s > 0$, fixed),

then $\frac{x_N}{f(N)}$ a gaussian random variable
for some scaling function $f(N)$.

Note, that the scaling function $f(N)$ is not required to be $N^{0.5}$ (normal diffusion). For example consider:

$$p(x) = \frac{A}{1+|x|^3}, \quad U(x) \sim \frac{1}{x} dx \sim \log(x)$$

$$\lim_x \frac{\log(sx)}{\log(x)} = 1 + \frac{\log(s)}{\log(x)} = 1$$

$$\text{if } \zeta_N = \frac{x_N}{\sqrt{N \log N}}, \text{ then } \phi_N(\zeta) = a \sqrt{N \log N} P_N(\zeta \sqrt{N \log N}) \sim \frac{e^{-\frac{\zeta^2}{2}}}{\sqrt{2\pi}}$$

So we get back the Gaussian distribution, however the half-width scales as $(N \log N)^{0.5}$. This example of anomalous diffusion represents a “superdiffusive” process. Essentially, the normal diffusive scaling breaks down before the Gaussian shape of the distribution breaks down for borderline cases.

3 Levy Distributions

3.1 Definition

A symmetric Levy distribution, denoted as $\ell(a, x)$ is a PDF with a characteristic function:

$$l_\alpha(a, k) = e^{-a|k|^\alpha}$$

[1]

Note that this can only be a valid characteristic function for $0 < \alpha \leq 2$ because the variance of $\ell(a, x)$ does not exist (i.e. $\langle x^2 \rangle = \infty$) for $\alpha > 2$. We have already seen some examples of Levy distributions. The Gaussian distribution is $\ell_2(\sigma^2/2, x)$:

$$p(k) = e^{-\frac{\sigma^2 k^2}{2}} \quad p(x) = \frac{e^{-\frac{x^2}{2\sigma^2}}}{\sqrt{2\pi\sigma^2}}$$

And the Cauchy distribution is $\ell_1(a, x)$:

$$p(k) = e^{-a|k|} \quad p(x) = \frac{a}{\pi(a^2 + x^2)}$$

Actually, these are the only two Levy distributions that can be inverted and expressed as elementary functions. We saw that the Cauchy distribution had infinite variance. In fact, for $0 < \alpha < 2$, all Levy distributions have infinite variance. Two important features of Levy distributions are: 1) They are stable under addition, i.e. a distribution of iid random Levy variables approaches a Levy distribution (Levy stable laws) and 2) the parameter a controls the tail amplitude, which we will see to be additive for N steps. As an aside, note that $\langle |x| \rangle = \infty$ for $\alpha \leq 1$.

3.2 Large x expansion

The large x asymptotic expansion for $l(a, x)$ is:

$$l_\alpha(a, x) \sim \frac{1}{\pi} \sum_{m=1}^{\infty} \frac{a^m \sin(\frac{\alpha m \pi}{2})}{m! |x|^{m\alpha}}, \quad |x| > a$$

This series diverges for $|x| > 1$ and converges (Laurent) for $|x| < 1$ where $|x| > a$.

At leading order, it reduces to:

$$l_\alpha(a, x) \sim \frac{\sin(\frac{\alpha \pi}{2})}{\pi} \frac{a}{|x|^{1+\alpha}}, \quad |x| > a$$

where we have a familiar power-law tail with amplitude a .

3.3 Small x expansion

We first simplify the inverse transform, realizing that only the even portion of e^{-ikx} contributes to the integral:

$$l_\alpha(a, x) = \int_0^\infty e^{-ikx} e^{-a|k|^\alpha} \frac{dk}{2\pi} \sim \frac{2}{\pi} \int_0^\infty \cos(kx) e^{-ak^\alpha} \frac{dk}{2\pi}$$

For small x, expand $\cos(kx)$:

$$l_\alpha(a, x) \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} (kx)^{2m} e^{-ak^\alpha} \frac{dk}{\pi}, \quad x \rightarrow 0 \quad [2]$$

Define the Gamma function:

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$$

and define a new variable, ω :

$$\omega = ak^\alpha, \quad d\omega = \alpha k^{\alpha-1} dk$$

This simplifies Equation 2 to:

$$l_\alpha(a, x) \sim \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} x^{2m} \int_0^\infty \frac{\omega^{\frac{2m}{\alpha}}}{a} e^{-\omega} \frac{1}{a\alpha} \frac{\omega^{\frac{\alpha-1}{\alpha}}}{a} \frac{d\omega}{\pi}, \quad x \rightarrow 0$$

$$\sim \frac{1}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m)!} \frac{x^{2m}}{a^{\frac{2m+1}{\alpha}}} \frac{2m+1}{\alpha}, \quad x \rightarrow 0$$

This series diverges for $|x| < 1$ and converges (Taylor series) for $1 \leq a \leq 2$ for $|x| < a$.

At leading order it reduces to:

$$l_\alpha(a, x) \sim \frac{1}{\pi \alpha a^{\frac{1}{\alpha}}}, \quad x \rightarrow 0$$

Thus $l_\alpha(a, x) \rightarrow \infty$ as $a \rightarrow 0$, so the center of the distribution gets sharper and higher and the tails get fatter. Figure 1 shows the Gaussian distribution, the Cauchy distribution, and a general Levy distribution as $a \rightarrow 0$.

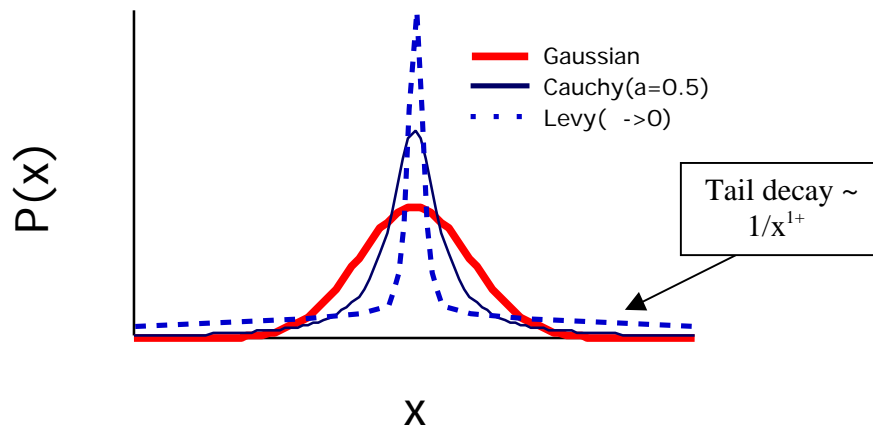


Figure. 1 Shapes of the Gaussian and Cauchy distributions compared with the Levy distribution as $a \rightarrow 0$.

In addition, note the asymptotic form of the Levy distribution as both $a \rightarrow 0$ and $x \rightarrow 0$:

$$l_\alpha(a, x) \sim \frac{1}{\pi \alpha} e^{-\frac{x^2}{2\sigma^2}}, \quad x \rightarrow 0 \quad \& \quad \alpha \rightarrow 0$$

$$\sigma = \sqrt{\frac{\pi}{2} \frac{(1 - 1/e)\alpha}{\sqrt{1/\alpha}}}$$

Thus, the central region of a Levy distribution in the low a limit looks like a Gaussian distribution with width $\sim a^{3/2}$.

4 Levy flights

Because of the possibility of taking “large” steps (due to the slow decay of the tails) we must address how time is measured during a random Levy process. We distinguish between a random “walk” and a random “flight.” Each step of a random walk takes a variable time dependent on the length of the step. For example, a random walk correctly models a random process at constant velocity. On the other hand, a random flight means that the time between both large and small random steps is constant. Here we consider Levy flights, realizing that the diffusive scaling for Levy walks will differ.

The position after N iid Levy steps is:

$$\vec{X}_N = \sum_{n=1}^N \vec{x}_n$$

We define:

$$p(x) = l_\alpha(a, x) \quad \text{PDF for } x_n$$

$$P_N(x) \quad \text{PDF for } N \text{ Levy displacements}$$

So:

$$l_\alpha(a, k) = e^{-a|k|^\alpha}$$

$$P_N(k) = e^{-a_N|k|^\alpha} \quad \text{where } a_N = Na$$

Now we solve for $P_N(x)$ and address how the width scales with N . First, take the inverse Fourier transform to solve for $P_N(x)$:

$$P_N(x) = \int_{-\infty}^{\infty} e^{-ikx} e^{-Na|k|^\alpha} \frac{dk}{2\pi} \quad [3]$$

Next we introduce ω and ζ :

$$\omega = kN^{\frac{1}{\alpha}} \quad |\omega|^\alpha = |k|^\alpha N$$

$$\zeta = \frac{x}{N^{\frac{1}{\alpha}}}$$

Substituting back into Equation 3 gives:

$$P_N(x) = \int_{-\infty}^{\infty} e^{-i\omega\zeta} e^{-a|\omega|^\alpha} \frac{d\omega}{2\pi} \frac{1}{N^{\frac{1}{\alpha}}}$$

which is a scaled Levy distribution:

$$P_N(x) = \frac{1}{N^{\frac{1}{\alpha}}} l_\alpha(a, \frac{x}{N^{\frac{1}{\alpha}}})$$

Most importantly, note that the width of the distribution $\sim N^{1/\alpha}$. So for $\alpha < 2$, we have $N^{1/\alpha} \gg N$, characteristic of a “superdiffusive” process.

5 Examples of Levy flights

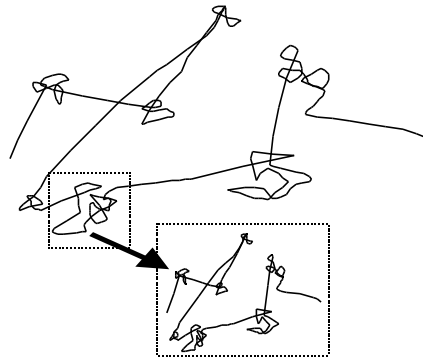


Figure 2. A characteristic Levy walk

Figure 2 shows a characteristic Levy flight for a low value of α . Note that the characteristic size of the system is the size of the largest step and that the flight is self-similar at higher magnifications.

5.1 Low density gases (“Nano”-systems)

One physical example of a Levy flight is a low-density gas. Here we consider a gaseous system where the characteristic system dimension (L) is much less than the mean free path of the gas (λ_{mfp}), $L \ll \lambda_{\text{mfp}}$. Note that $\lambda_{\text{mfp}} \sim 10^{-7}$ m at STP. The dimensionless parameter to consider here is the Knudsen number, $\text{Kn} = \lambda_{\text{mfp}}/L$, so $\text{Kn} \gg 1$ in these nano-systems.

In the $\text{Kn} \gg 1$ regime, the gas molecules essentially travel in straight lines and collisions with the system walls are dominant. Here we assume the wall is rough so that a colliding molecule rebounds at a random angle. Figure 3 depicts high Kn transport in a rough container.

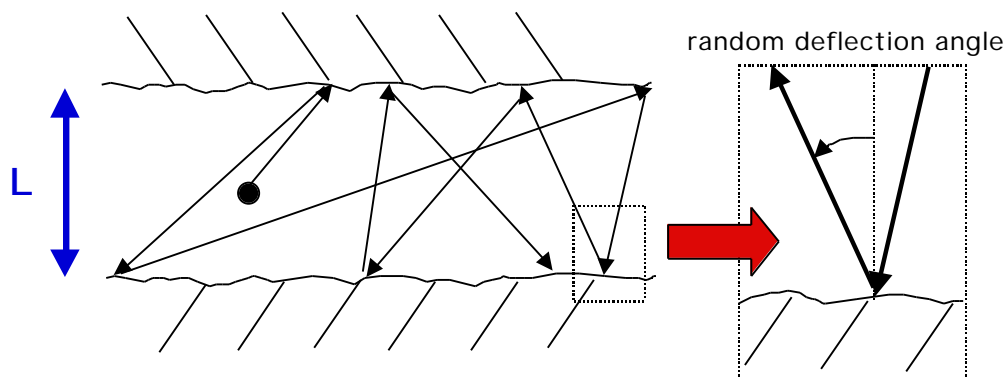
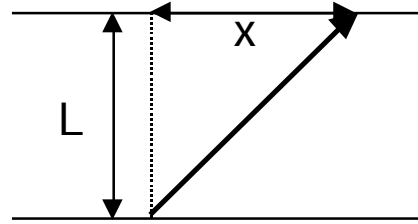


Figure 3. Transport of a gas with $\text{Kn} \gg 1$ in a rough-walled contained

We can use a geometric argument to determine the distribution of step sizes (x):

$$\begin{aligned}\tan \theta &= \frac{x}{L} \\ \frac{d\theta}{\cos^2 \theta} &= \frac{dx}{L} \\ d\theta &= \frac{L^2}{x^2 + L^2} \frac{dx}{L}\end{aligned}$$



The distribution of rebounding angles is uniform so:

$$\begin{aligned}p_\theta &= \frac{1}{\pi} \\ p_\theta d\theta &= p_x dx \\ p_x &= \frac{L}{x^2 + L^2} \frac{1}{\pi}\end{aligned}$$

Therefore the distribution of N steps is the Cauchy distribution with amplitude L .

$$P_N(x) = \frac{1}{N} l_1\left(L, \frac{x}{N}\right)$$

Note that this distribution is only valid for displacements smaller than the mean free path. The distribution width grows $\sim N$, so at large enough N diffusive behavior will return. Therefore, this nano-system may be modeled as a “truncated Levy flight” (cf. Fig. 4).

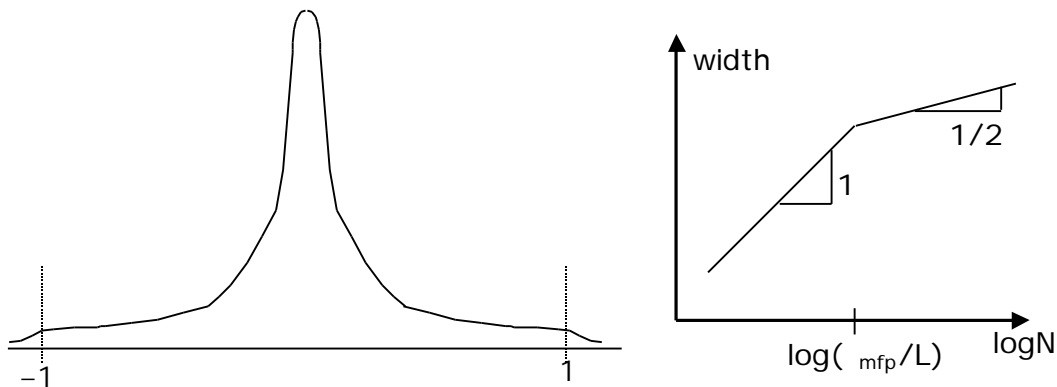


Figure 4. A truncated Levy flight. Note the transition from molecular transport (Cauchy) to diffusive transport (Gaussian).

5.2 Financial time series

Financial time series can also be modeled with a truncated Levy flight where the cutoff can be applied analytically. For more information, refer to Bouchard and Potters.

5.3 Polymer surface adsorption

A polymer adsorbed to a surface has a finite number of contact points with the surface that can be modeled as a Levy flight (Bouchard and George, 1998) (cf. Fig. 5).

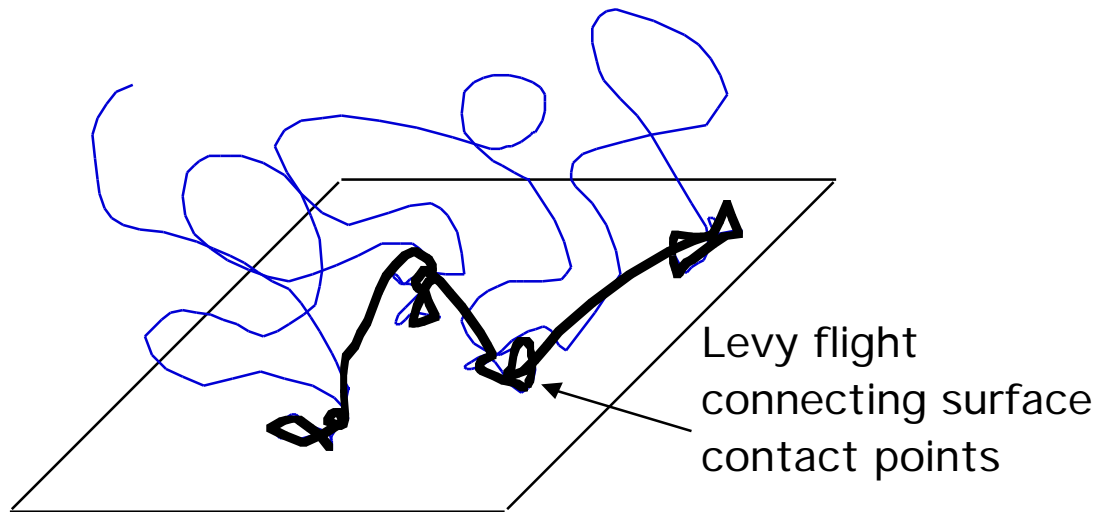


Figure 5. Polymer adsorbed to a surface