Lecture 10: Persistent Random Walks and the Telegrapher's Equation

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1 Review from last lecture

To review from last lecture, we've considered a random walk with correlated displacements:

$$\vec{X}_N = \sum_{n=1}^{N} \Delta\vec{x}_n, \qquad <\Delta\vec{x}_n \quad \Delta\vec{x}_{n+m}> = C(m)$$

where C(m) is the correlation function. As an example, we considered an exponentially decaying correlation function:

$$C(m) = \rho^m = e^{\frac{-2m}{n_c}} \quad where \;\; n_c = \frac{2}{-\log\rho}$$

We found:

$$\frac{<\vec{X}_N{}^2>}{\sigma^2} = \frac{1+\rho}{1-\rho}N + 2\rho\frac{\rho^N - 1}{(\rho-1)^2}$$

We saw the transition from ballistic to diffusive behavior by scaling and taking the limit ρ→1:

$$\frac{<\vec{X}_N{}^2>^{1/2}}{n_c\sigma} = \sqrt{\frac{N}{n_c} + \frac{1}{2}(e^{-2N/n_c} - 1)}$$

$$\sim \frac{N}{n_c} \quad for \; \frac{N}{n_c} << 1 \qquad \text{ballistic}$$

$$\sim \sqrt{\frac{N}{n_c}} \quad for \; \frac{N}{n_c} >> 1 \qquad \text{diffusive}$$

2 Persistent Random Walk: Definition

Here we consider the Persistent Random Walk example, a *correlated random walk* on a hypercubic lattice in which the walker has probability α of continuing in the same direction as the previous step. In d dimensions, the walker has probability (1-α)/(2d-1) of going any other direction. Today, we specifically consider the d=1 example (note that it is non-trivial to scale up to higher dimensions).

Figure 1 shows a diagram of the one-dimensional persistent random walk. The walker (at position 0), is about to take step n after taking step n-1.

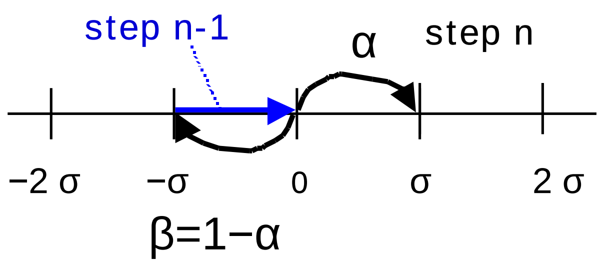

Figure 1. The persistent random walk (d=1)

The walker's displacements are correlated as discussed last lecture. We determine the correlation coefficient, ρ:

$$\rho = \frac{< \Delta x_{n+m} \Delta x_n >}{\sigma^2} = \alpha - (1-\alpha) = 2\alpha - 1$$

Note that if α=1/2 then ρ=0 and we have an uncorrelated iid random walk. If α=1 then ρ=1 and we have a perfectly correlated random walk (always moving same direction). If α=-1 then ρ=-1 and we have an anti-correlated random walk (hopping between two sites).

We can already predict from the exponentially decaying correlation example that we will eventually get diffusive scaling:[1]

$$\frac{D(\alpha)}{D_o} = \frac{1+\rho}{1-\rho} = \frac{\alpha}{\beta} \quad where \quad D_o = \frac{\sigma^2}{2\tau}$$

Our goal will be to solve for the probability distribution function $P_n(m)$ of the displacement *m* after *n* steps for the persistent random walk.

[1] Note that in higher dimensions (d>1) $D_o=\sigma^2/2d\tau$.

3 Difference Equations

We will solve for $P_n(m)$ by use of difference equations, a technique introduced by M. Kai. First, we decompose $P_n(m)$ into two probabilities: $A_n(m)$ for the walker to end up at *m* after *n* steps coming from the left of *m* and $B_n(m)$ for the walker to end up at *m* after *n* steps coming from the right of *m*.

$$A_n(m) = \mathrm{Pr}ob(X_n = m \mid X_{n-1} = m-1)\ \mathrm{Pr}ob(X_{n-1} = m-1)$$
$$B_n(m) = \mathrm{Pr}ob(X_n = m \mid X_{n-1} = m+1)\ \mathrm{Pr}ob(X_{n-1} = m+1)$$

$$P_n(m) = A_n(m) + B_n(m)$$

Let α be the transition probability for the random walker to move in the same direction as the previous step. Let β (=1-α) be the transition probability for reversing direction. We can decompose the system into two situations (coming from left vs. right at step n), and for each of these possibilities, there are two ways to arrive at *m* (depends on direction of step n-1).

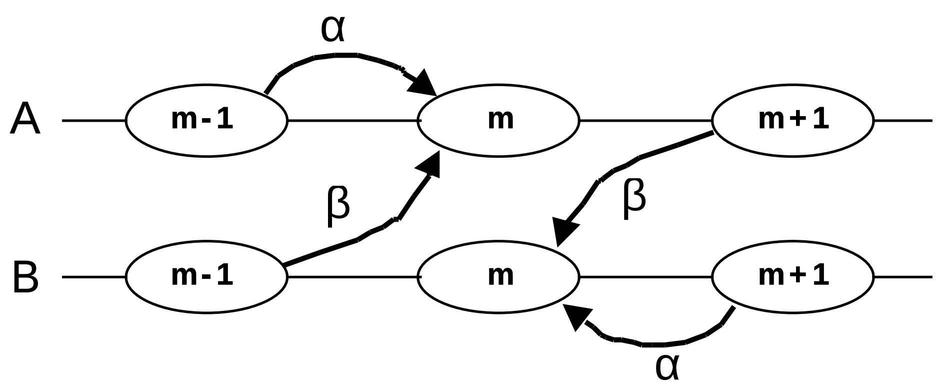

Figure 2. A split probability diagram of the persistent random walk. Continuing in the same direction maps to horizontal movement on this diagram, whereas reversing direction corresponds to vertically jumping from A to B (or vice versa).

Thus the difference equations for the probabilities $A_{n+1}(m)$ and $B_{n+1}(m)$ are:

$$A_{n+1}(m) = \alpha\, A_n(m-1) + \beta\, B_n(m-1)$$
$$B_{n+1}(m) = \beta\, A_n(m+1) + \alpha\, B_n(m+1)$$

The most useful property of decomposing $P_n(m)$ and using the difference equations is that if the previous state $[A_n(m),B_n(m)]$ is known, then the next state can be determined. Thus the evolution of $[A_n(m), B_n(m)]$ is a Markov chain. Conversely, directly using the probability $P_n(m)$ requires knowledge of two previous time steps.

To solve for $[A_n(m), B_n(m)]$ we can either solve the problem exactly with a DFT or we can find an asymptotic solution by taking the continuum limit. We will discuss the latter, which will yield a partial differential equation in the limit $n \to \infty$.

3.1 Continuum limit approximation

Let's consider continuous variables as $n \to \infty$:

$$
\begin{aligned}
a(m\sigma, n\tau) &\equiv A_n(m) \\
b(m\sigma, n\tau) &\equiv B_n(m) \\
p(m\sigma, n\tau) &\equiv P_n(m) = A_n(m) + B_n(m) \\
q(m\sigma, n\tau) &\equiv Q_n(m) = A_n(m) - B_n(m)
\end{aligned}
$$

The function q has been introduced which encodes the effect of correlations. Note that $q=0$ when $\alpha=1/2$ (uncorrelated) and q is non-zero otherwise.

Taylor expand the difference equations around $x=m\sigma$ and $t=n\tau$:

$$
a + \tau a_t + \frac{\tau^2}{2} a_{tt} + \ldots = \alpha \left(a - \sigma a_x + \frac{\sigma^2}{2} a_{xx} - \ldots \right) + \beta \left(b - \sigma b_x + \frac{\sigma^2}{2} b_{xx} - \ldots \right)
$$

$$
b + \tau b_t + \frac{\tau^2}{2} b_{tt} + \ldots = \beta \left(a + \sigma a_x + \frac{\sigma^2}{2} a_{xx} + \ldots \right) + \alpha \left(b + \sigma b_x + \frac{\sigma^2}{2} b_{xx} + \ldots \right)
$$

Then add these equations, using $\alpha+\beta=1$, $a+b=p$, $a-b=q$, to get:

$$
\tau p_t + \frac{\tau^2}{2} p_{tt} + \ldots = (\beta - \alpha)\sigma q_x + \frac{\sigma^2}{2} p_{xx} + \ldots \qquad (1)
$$

Also subtract the two expansions to get:

$$
q + \tau q_t + \frac{\tau^2}{2} q_{tt} + \ldots = (\alpha - \beta) q - \sigma p_x + \frac{(\alpha - \beta)}{2} \sigma^2 q_{xx} + \ldots \qquad (2)
$$

Take the x-derivative of Equation 2 :

$$
(1 - \alpha + \beta) q_x + \tau q_{tx} + \ldots = -\sigma p_{xx} + \ldots \qquad (3)
$$

and substitute this expression for q_x into Equation 1 to arrive at:

$$
p_t + \frac{\tau}{2} p_{tt} + \ldots = \frac{(\beta - \alpha)}{\tau} \sigma \left(\frac{-\sigma p_{xx}}{1 - \alpha + \beta} - \frac{\tau q_{tx}}{1 - \alpha + \beta} \right) + \frac{\sigma^2}{2\tau} p_{xx} + \ldots \qquad (4)
$$

It will matter how we take the continuum limit to see which terms are important.

3.2 Diffusive scaling

Here we take the continuum limit (as $n\to\infty$, $\sigma\to 0$ & $\tau\to 0$) with $D_o=\sigma^2/2\tau$ remaining constant. Going back to the x-derivative equation (Eq. 3), in this limit and using $\beta=1-\alpha$:

$$q_x \sim \frac{-\sigma}{2\beta} p_{xx} \tag{5}$$

Substituting this back into Equation 1 and taking the limits $\sigma\to 0$ and $\tau\to 0$ yields a diffusion equation with a modified diffusion coefficient:

$$p_t = D_o\left(1+\frac{\alpha-\beta}{\beta}\right)p_{xx} + O(\tau) \tag{6}$$

where

$$\frac{D}{D_o} = 1+\frac{\alpha-\beta}{\beta} = \frac{\alpha}{\beta} = \frac{\alpha}{1-\alpha} = \frac{1+\rho}{1-\rho}$$

as expected for an exponentially decaying correlation function.

3.3 Ballistic scaling

Here we will take the continuum limit in a different way in order to probe the transition from persistence ("ballistic scaling") to diffusive scaling. We again take the limit as $n\to\infty$, $\sigma\to 0$ & $\tau\to 0$, however we now require $v=\sigma/\tau$ to remain fixed.

We further take the strong persistence limit with $\alpha\to 1$ and $\rho\to 1$:

$$\alpha = 1-\frac{\varepsilon}{2}, \quad \beta = \frac{\varepsilon}{2} \qquad \rho = \alpha-\beta = 1-\varepsilon$$

The scaling of ε with σ and τ is still to be determined.

Returning to Equation 3 we see it simplifies to:

$$q_x \sim -\frac{\tau}{\varepsilon} q_{tx} - \frac{\sigma}{\varepsilon} p_{xx} \tag{7}$$

Now we introduce a characteristic timescale for ballistic motion $\tau_c=\tau/\varepsilon$. Applying this to Equation 7 gives:

$$q_x \sim -\tau_c q_{tx} - v\tau_c p_{xx} \tag{8}$$

Now we return to Equation 1, and in this limit we find:

$$p_t \sim (\beta - \alpha)\frac{\sigma}{\tau} q_x = -(1-\varepsilon) v q_x \tag{9}$$

Substitute for the q_{tx} term in Equation 8 by taking the time derivative of Equation 9:

$$p_{tt} \sim -(1-\varepsilon) v q_{tx} \sim -v q_{tx} \tag{10}$$

Inserting Equations 9 & 10 into Equation 8 yields the **telegrapher's equation**:

$$\boxed{p_{tt} + \frac{1}{\tau_c} p_t = v^2 p_{xx}} \tag{11}$$

Which is at leading order for $n \to \infty$, ($\sigma \to 0$ & $\tau \to 0$ approaching with $\sigma/\tau = v =$ constant), and $\alpha \to 1$ (approaching as $\alpha = 1 - \tau/2\tau_c$.)

For times much smaller than τ_c, the telegrapher's equation reduces to the wave equation; at times much longer than τ_c, it reduces to the diffusion equation. Thus it correctly models a signal which moves initially as a wave (Fig. 3A), but over time decays due to noise (Fig. 3B).

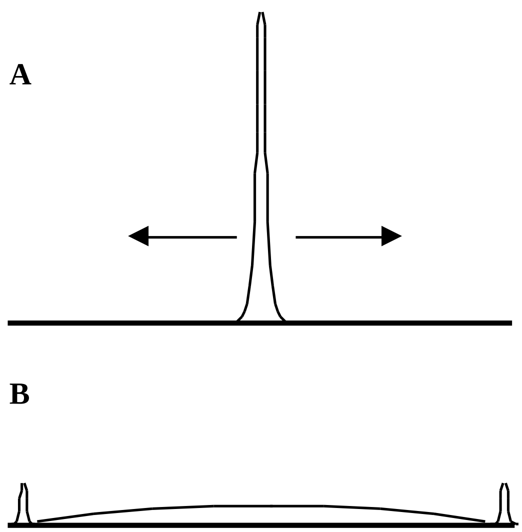

Figure 3. A) Wave motion of a signal modeled by the telegrapher's equation **B**) Diffusive motion of a signal modeled by the telegrapher's equation.