Problem Set Number 07, (18.353/12.006/2.050)j
MIT (Fall 2024)

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Due: Tuesday December 3, 2024 (turn it in via the canvas course website).

Important. This problem set has two parts (function explained below). Notice that the "regular problem set" is shorter than usual. There will be another problem set (also shorter) due on the last day of lectures. I want to have 8 problem sets, so then I can *drop your "worse" problem from the first 6* (i.e.: the ones that you already did).

Regular problem set. This "is" problem set #1 and it applies to everyone, as usual.

Recovery problems. This applies **only** to students that have two problems with less than 40/60 within the first 6 already assigned. The worse one will be dropped for the grading (as explained above). I will then use the result of the three problems below to lift the grade for the other one by 12 points (or less).

Contents

Part I
Regular problem set

1 Homoclinic bifurcation with a computer #01

Statement: Homoclinic bifurcation with a computer #01

Consider the system $$\dot{x} = \mu\, x + y - (2\,\mu + 1)\, x^2 \quad \text{and} \quad \dot{y} = -x + \mu\, y + (2 - \mu)\, x^2, \qquad (1.1)$$
where μ is a parameter. For this system:

(1) Find and analyze/classify the critical points.

(2) Using a computer, show that a supercritical Hopf bifurcation happens at $\mu = 0$.

(3) Using a computer, show that an homoclinic bifurcation happens for $0 < \mu = \mu_c$, where μ_c is very small.
You can get a rough idea of "how small is small" by recalling that the "radius" of the Hopf bifurcation limit cycle grows like the square root of the parameter deviation from the onset. Hence, in this case it should be $C\,\sqrt{\mu_c} \approx$ distance from the origin to the saddle, where C is a constant. Of course, you do not know C, so this estimate can be off by some factor, probably less than 10. **Do this estimate before doing (3).**

(4) Find the approximate value of μ_c.

(5) Illustrate your results with phase plane plots in the region $-0.4 < x,\, y < 0.6$.

Hint: The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence *check the behavior of the orbits near the critical point for $\mu = 0$, to see if the nonlinear terms stabilize or de-stabilize.*

Suggestions for computation. Use **PHPLdemoB2**, which allows for easy parameter searches. To get precisely controlled plots, you can use **PHPLplot** or **PHPLplot_v2**, which are a bit less user friendly, but allow precise control. Note that, for μ small, the radial dynamics near the origin is slow, thus you may need "large" integration times … **do not use the default integration times and tolerances in the scripts.**

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2 Newton's method in the complex plane #02

Statement: Newton's method in the complex plane #02

Suppose that you want to solve an equation, $g(x) = 0$. Then you can use *Newton's method*, which is as follows: Assume that you have a "reasonable" guess, x_0, for the value of a root. Then the sequence $x_{n+1} = f(x_n)$, $n \geq 0$, where

$$f(x) = x - \frac{g(x)}{g'(x)}, \tag{2.1}$$

converges (very fast) to the root.

Remark 2.1 (The idea). Assume an approximate solution $g(x_a) \approx 0$. Then write $x_b = x_a + \delta x$ to improve it, where δx is small. Then $0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a)\,\delta x \;\Rightarrow\; \delta\,x \approx -\frac{g(x_a)}{g'(x_a)}$, and (2.1) follows.
Of course, if x_0 is not close to a root, the method may not converge. Even if it converges, it may converge to a root far away from x_0, not necessarily the closest root. In this problem **we investigate the behavior of Newton's method in the complex plane, for arbitrary starting points.** ♣

Consider iterations of the map generated by Newton's method for the roots of $z^4 - 1 = 0$. i.e.:

$$z_{n+1} = f(z_n) = \left(\frac{3}{4} + \frac{1}{4\,z_n^4}\right) z_n, \quad n \geq 0, \tag{2.2}$$

where $0 < |z_0| < \infty$ is arbitrary, and the z_n are **complex numbers.**
Note that

$$\zeta_1 = 1, \quad \zeta_2 = e^{i\,\pi/2} = i, \quad \zeta_3 = e^{i\,\pi} = -1, \quad \text{and} \quad \zeta_4 = e^{i\,3\,\pi/4} = -i, \tag{2.3}$$

are the roots of $z^4 = 1$.

Your tasks: Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every initial point z_0, draw a colored dot at the position of z_0, where **the colors are picked as follows:**

$$z_n \to \zeta_1, \ \textbf{green.} \quad z_n \to \zeta_2, \ \textbf{red.} \quad z_n \to \zeta_3, \ \textbf{blue.} \quad z_n \to \zeta_4, \ \textbf{yellow.} \quad \textbf{No convergence, black.} \tag{2.4}$$

What do you see? Do blow ups (see item h6 below) of the limit regions between zones.

Hints, practical numerical considerations, etc.

h1. Divide the region [I *strongly* suggest the square $-2 \leq \text{Re}(z_0)$, $\text{Im}(z_0) \leq 2$] where the initial data z_0 will be picked into pixels, then pick a z_0 at the center of each pixel, and color the pixel according to (2.4).

h2. If you use MatLab, **do not plot points.** As suggested in item **h1**, plot pixels — use the command image$(x,\, y,\, C)$ to plot, where: $x = \text{Re}(z_0)$, $y = \text{Im}(z_0)$, and C encodes the color. **Why?** Because using points leaves a lot of unpainted space in the figure, and **gives huge file sizes** if you use enough pixels to get a good picture.

h3. Deciding convergence. Deciding that the sequence converges is easy: once z_n gets "close enough" to one of the roots, then the very design of Newton's method guarantees convergence. Thus, given a z_0, compute z_N for some large N, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some "small" tolerance δ — which does not have to be very small, but pick $\delta = 10^{-5}$ for extra caution. If this criteria is not satisfied for any of the roots, then classify the sequence starting at z_0 as "non-convergent".

You can get reasonable pictures with $N = 50$ iterations on a 150×150 grid — a larger N is needed when refining near the boundary between zones. For the answer I used a 500×500 grid and $N = 100$ iterations — which I increased to $N = 150,\ 200,\ 250$ for the blow ups of details (likely over-kill).

h4. Compute in parallel. If you use MatLab, make sure to do all the sequences (one for each pixel) in parallel, using vector/matrix operations. This is much, much, faster than a "for loop".

h5. Avoid division by zero. Note that (2.2) ceases to make sense if $z_n = 0$ — classify this as non-convergence. This can cause a problem if you are computing all the sequences in parallel, because this requires all of them to be computed from z_0 to z_N. One way to get around this (in MatLab) is as follows: Place all the iterates in a complex matrix **Zn**, where the entry $(p,\ q)$ corresponds to z_n for the sequence starting in the $(p,\ q)$ pixel. Then, before computing the next iterate, execute: **Zn = Zn + del*(Zn == 0)**, where **del = 1e-20**. † After this sequences with $z_n = 0$ will produce a very large z_{n+1}, which is guaranteed not return to the vicinity of the roots ζ_j for many iterations (more than 300), resulting in "effective" non-convergence. ‡

† This replaces zero entries in **Zn** by **del**, because the logical operator **(Zn == 0)** yields zero for all non-zero entries in **Zn**, and one for zero entries.

‡ The result will be $z_{n+1} \approx (1/4)10^{60}$, while for z_n large (2.2) reduces to $z_{n+1} \approx (3/4)\, z_n$. Hence returning to $z_{n+M} = O(1)$ requires, roughly, $(3/4)^M\, 10^{60} = O(1)$.

h6. Regions to explore. Do (at least) four figures, exploring the regions: $-2 \le x,\ y \le 2$, $-0.3 \le x,\ y \le 0.7$, $-0.39 \le x,\ y \le 0.46$, and $-0.432 \le x,\ y \le 0.436$, where $x = \mathrm{Re}(z_0)$ and $y = \mathrm{Im}(z_0)$.

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3 Quasiperiodic functions and Lissajous figures

Statement: Quasiperiodic functions and Lissajous figures.

Task #1. Using a computer, plot the curve whose parametric equations are $x(t) = \sin(t)$ and $y(t) = \sin(\omega\, t)$, for the following rational and irrational values of the parameter ω.

$$\left.\begin{array}{lll} \textbf{(1)}\ \ \omega = 3 & \textbf{(2)}\ \ \omega = \dfrac{2}{3} & \textbf{(3)}\ \ \omega = \dfrac{5}{3} \\ \textbf{(4)}\ \ \omega = \sqrt{2} & \textbf{(5)}\ \ \omega = \pi & \textbf{(6)}\ \ \omega = \dfrac{1}{2}\left(1+\sqrt{5}\right) \end{array}\right\} \text{Note: (6) is the "golden ratio"} \tag{3.1}$$

Task #2. Do the same for $x(t) = \sin(t)$ and $y(t) = \cos(\omega\, t)$.

The resulting **curves are called Lissajous figures.** They can be displayed on an oscilloscope by using two ac signals of different frequencies as inputs. Note that you will see three qualitatively different types of curves: In the periodic case (i.e.: ω is rational) (i) open curves and (ii) closed curves; and in the true quasi-periodic case (i.e.: ω is irrational) (iii) dense curves in the plotting region.

Task #3, Optional. Can you explain what triggers the difference between (i) and (ii)?

Task #4. Add $\omega = 1$ and $\omega = 1.02$ for the case $x(t) = \sin(t)$ and $y(t) = \cos(\omega\, t)$. Describe what you see.

Task #5 (this task requires no answer). Run the two scripts [QuasiPer2sincos and QuasiPer2sinsin] provided with the problem statement. These two scripts illustrate two important properties of quasi-periodic functions: (1) The "natural" phase space for them is a Torus with as many dimensions as periods they have; and (2) They have "almost periods" $T_n > 0$ over which they repeat with arbitrarily small errors.

Remark 3.2 Obviously, in a numerical calculation ω cannot be "irrational". However, an irrational ω will be approximated by a rational number which is the quotient of two very large integers. The result will be a periodic curve with a very, very, complicated and a very, very long period.

4 Sierpinski's carpet

Statement: Sierpinski's carpet

Consider the process shown in figure 4.1. Divide the closed unit box into nine equal boxes, and delete the open central box. Repeat for each of the eight remaining sub-boxes, and so on. Figure 4.1 shows the first two stages. **Tasks: A.** Sketch the next stage, S_3. **B.** Find the similarity dimension of the limiting fractal, known as the **Sierpinski**

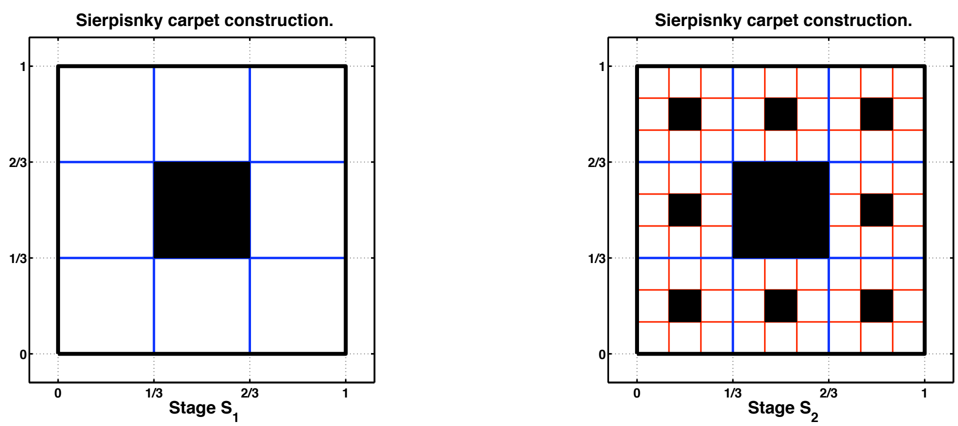

Figure 4.1: **Sierpinski's carpet construction.** The areas shaded in black are the parts of the original square deleted.

carpet. C. Show that the Sierpinski carpet has zero area.

Part II
Recovery problems

5 Index for a critical point with zero determinant

Statement: Index for a critical point with zero determinant

Consider a phase plane system

$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{5.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin $\mathcal{O}$ is an isolated critical point. That is $f(0,\,0) = g(0,\,0) = 0$, and there are no solutions to $f(x,\,y) = g(x,\,y) = 0$ with $0 < x^2 + y^2 < \epsilon$ — for some ϵ.

2. Let A be the 2×2 matrix corresponding to the linearized system near $\mathcal{O}$, with $\tau = \text{tr}(A)$ and $\Delta = \det(A)$. Suppose that $\boldsymbol{\Delta = 0}$ and $\boldsymbol{\tau > 0}$ — so that one eigenvalue of A vanishes, and the other equals τ.

This is a structurally unstable situation, in particular: the index for $\mathcal{O}$ is not determined at all by the linearized equations. **Construct examples of the above situation where:**

A. $\mathcal{I} = \text{index}(\mathcal{O}) = 1$.

B. $\mathcal{I} = \text{index}(\mathcal{O}) = -1$.

C. $\mathcal{I} = \text{index}(\mathcal{O}) = 0$.

Sketch the phase plane portraits for the systems that you construct.

Hints. Consider the linear system $\dot{Y} = AY$, and then add a nonlinear correction which:

For part **A**. Makes $\mathcal{O}$ into a (nonlinear) node.

For part **B**. Makes $\mathcal{O}$ into a (nonlinear) saddle.

For part **C**. Makes $\mathcal{O}$ into a (nonlinear) saddle on one side, and a (nonlinear) node on the other.

Note that you can do **A**-**B** by writing a linear system $\dot{Y} = \tilde{A}Y$ where $\tilde{A}$ gives you either a node or a saddle, and then replacing the entries that vanish in A by appropriate nonlinear terms that vanish at the origin. **C** is a little more involved, so first do **A**-**B**.

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6 Limit cycle bifurcation with a computer #01

Statement: Limit cycle bifurcation with a computer #01

The system
$$\boldsymbol{\dot{x} = y/\delta} \quad \text{and} \quad \boldsymbol{\dot{y} = (2\,\mu\,y - (1+\mu^2)\,x + x^2\,(\mu\,x - y))/\delta}, \tag{6.1}$$
where $\boldsymbol{\delta = 1 + x^2}$, undergoes a supercritical Hopf bifurcation at $\boldsymbol{\mu = 0}$. Beyond that, as $\boldsymbol{\mu}$ increases, while the critical points at $\boldsymbol{(x, y) = (\pm x_0,\, 0)}$ ($\boldsymbol{x_0 = \sqrt{\mu + 1/\mu}}$) move towards the origin. For $\boldsymbol{\mu = \mu_c}$, $\boldsymbol{0.3 < \mu_c < 0.4}$, **another bifurcation occurs.** Using a computer:

(1) Describe what happens at $\mu = \mu_c$.

(2) Find the nature of all the critical points of the system.

(3) Ascertain the value of μ_c with two significant digits.

Illustrate your conclusions with three phase plane portraits in the region $-3 < x,\ y < 3$, one for $\mu < \mu_c$, one for $\mu > \mu_c$, and one for $\mu \approx \mu_c$.

Remark #1. Note that the system has the symmetry $\boldsymbol{(x,\,y) \to (-x,\,-y)}$.

Suggestions for computation. Since the problem involves investigating a situation as a parameter varies, with an equation with many terms, PHPLdemoB is not the best tool (impractical). Instead, I suggest that you use **PHPLdemoB2**, which will allow you to do $\boldsymbol{\mu}$ "sweeps" efficiently. Once you know well what happens, to produce "neat" final plots, you could use **PHPLplot** or **PHPLplot_v2**, which are less user friendly than PHPLdemoB2, but allow precise control of the orbits.

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7 Lorenz equations: linear stability for two CP

Statement: Lorenz equations: linear stability for two CP

Consider the **Lorenz equations,** $\dot{\boldsymbol{x}} = \boldsymbol{\sigma}\,(\boldsymbol{y} - \boldsymbol{x})$, $\quad \dot{\boldsymbol{y}} = \boldsymbol{r}\,\boldsymbol{x} - \boldsymbol{y} - \boldsymbol{x}\,\boldsymbol{z}$, $\quad \dot{\boldsymbol{z}} = \boldsymbol{x}\,\boldsymbol{y} - \boldsymbol{b}\,\boldsymbol{z}$, where $\boldsymbol{\sigma},\ \boldsymbol{b} > \boldsymbol{0}$ and $\boldsymbol{r} > \boldsymbol{1}$.

1. Show that the fixed points for the Lorenz equations are: $\boldsymbol{C}^{\pm} = (\pm\sqrt{\boldsymbol{b}\,(\boldsymbol{r}-\boldsymbol{1})},\ \pm\sqrt{\boldsymbol{b}\,(\boldsymbol{r}-\boldsymbol{1})},\ \boldsymbol{r}-\boldsymbol{1})$ and the origin. Note that if $\boldsymbol{r} \leq \boldsymbol{1}$ the only fixed point is the origin.
2. Show that the characteristic polynomial for the eigenvalues of the Jacobian matrix at $C^{\pm}$ is
$$\lambda^3 + (\sigma + b + 1)\,\lambda^2 + (r + \sigma)\,b\,\lambda + 2\,b\,\sigma\,(r - 1) = 0.$$
3. By seeking solutions of the form $\lambda = i\,\omega$, with ω real and non-zero, show that there is a pair of purely imaginary eigenvalues when
$$r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right), \quad \text{provided that } \sigma > b + 1.$$
The value $\boldsymbol{r_H}$ is where a subcritical Hopf bifurcation occurs. *Explain why we need to assume* $\sigma > b + 1$.

THE END.