Problem Set Number 06, (18.353/12.006/2.050)j
MIT (Fall 2024)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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Due: Thu. Nov. 21, 2024 (turn it in via the canvas course website).

Contents

1 Degenerate bifurcation - fails to be Hopf

Statement: Degenerate bifurcation - fails to be Hopf

Consider the damped/driven Duffing oscillator

$$\ddot{x} + \mu\dot{x} + x - x^3 = 0, \quad (1.1)$$

where μ is a constant.

a) Show that the origin changes from a stable to an unstable spiral as μ decreases through zero.

b) Plot the phase portraits for $\mu > 0$, $\mu = 0$ and $\mu < 0$, and show that the bifurcation at $\mu = 0$ is a degenerate version of the Hopf bifurcation. In fact, **show that no periodic orbits are possible for $\mu \neq 0$.**
The nonlinearity is neither stabilizing, nor destabilizing. A "degenerate", very special, case. Hint: Energy/Liapunov.

2 Dog and duck in a pond

Statement: Dog and duck in a pond

A dog at the center of a circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight towards it. In other words, the dog's velocity vector always lies along the line connecting it to the duck. Meanwhile, the duck takes evasive action by swimming around the circumference as fast as it can, always moving counterclockwise.

Task #1. Assume that the pond has unit radius, and that both animals swim at the same constant speed, and **derive a pair of differential equations for the dog path.** Use a system of coordinates with the origin at the pond's center — see figure 2.1, and write equations for $\frac{dR}{d\theta}$ and $\frac{d\phi}{d\theta}$.
Notation: (i) R is the *distance between the dog and the duck*, **(ii)** θ is the polar angle for the duck's position, and **(iii)** ϕ is the angle (measured counter-clockwise) between the lines connecting the duck's position to the center of the pond, and the line connecting the duck's position to the dog's position. See figure 2.1. **Further: (iv)** let $v > 0$ be the *constant linear speed of the dog*, **(v)** let $s > 0$ be the *constant angular speed of the duck* — hence $\theta = s\,t + \theta_0$, for some constant θ_0, and **(vi)** let $k = v/s$ be the *ratio of the dog's speed to the duck's speed.*
The independent variable in the equations will be the angle θ, not time. However, because $\theta = s\,t + \theta_0$, θ and t are equivalent.

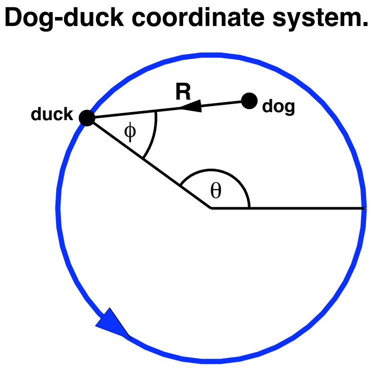

Figure 2.1: Dog and Duck in a pond
The Dog-Duck coordinate system; with the variables $\boldsymbol{R}$, $\boldsymbol{\phi}$, and $\boldsymbol{\theta}$.

Hint #1. Because the motion occurs in the plane, using complex numbers to represent vectors is useful. Thus, let $\boldsymbol{D = D(t)}$ = *position of the dog* at time t, and let $\boldsymbol{d = d(t) = e^{i\,\theta}}$ = *position of the duck* at time t. The *vector connecting the dog to the duck* is then $\boldsymbol{d - D = R\,e^{i\,\psi}}$, where ψ can be easily written in terms of θ and ϕ.

Hint #2. Notice that, along the edge of the pond, the swimming direction for the dog would always be towards the inside of the pond, except when/if the dog catches the duck (where ϕ is no longer defined, and the dog-duck coordinate system is singular). Thus *the dog stays inside the pond till/if it captures the duck, with* $\boldsymbol{-\pi/2 < \phi < \pi/2}$. This information will be useful when answering the questions below.

Task #2. Answer the question: what value does ϕ approach when/if the dog captures the duck? That is, as $R \downarrow 0$?
Hint #3. Look at the equation for $\frac{\mathbf{d}\boldsymbol{\phi}}{\mathbf{d}\boldsymbol{\theta}}$.

Task #3. Show that there is a critical $\boldsymbol{k_c > 0}$ such that: (i) If $\boldsymbol{k > k_c}$ the *dog catches the duck in a finite time.* (ii) If $\boldsymbol{k < k_c}$, the *dog never catches the duck.* Note that k_c is a simple number, the same one an intuitive argument suggests — the problem, however, requires more than an intuitive argument.
Hint #4. The case $k > k_c$ is easy to analyze; just look at the equation for $\frac{\mathbf{d}\boldsymbol{R}}{\mathbf{d}\boldsymbol{\theta}}$.
To show that there can be no capture for $k < k_c$, use the result in *Hint #2.*

Task #4 (optional beyond finding and clasifying the critical point(s)). Give a complete description of the phase plane (ϕ, R) for the case $0 < k < k_c$. In particular, show that there is a critical point that is a global attractor, corresponding to a final situation where the dog swims on a circle of radius k, at a constant distance from the duck. A complete answer here is challenging; you will have to: (i) use Dulac's criterion to rule out limit cycles, and (ii) construct trapping regions and use the Poincaré-Bendixon theorem to show that all the orbits end up at the critical point.

Task #5 (optional challenge question). What happens for $\boldsymbol{k = k_c}$?

3 Hopf bifurcation using a computer #02

Statement: Hopf bifurcation using a computer #02

For the following system

$$\frac{dx}{dt} = \mu\,x + y - x^3 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\,y + 2\,y^3, \tag{3.1}$$

a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, **plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.**

Hints: (a) The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence *check the behavior of the orbits near the critical point for a very small value of μ to see if the nonlinear terms stabilize or de-stabilize.* **(b)** In the *subcritical case* the limit cycle is unstable, very hard to compute forward in time. However, if you *run the system backwards in time the solutions converge to the limit cycle.* Note that the PHPL scripts include computations both backards and forwards in time.

4 Multiple scales and limit cycles #02

Statement: Multiple scales and limit cycles #02

Consider the nonlinear oscillator described by the equation

$$\frac{d^2x}{dt^2} - \epsilon\,(1-x^2)\,\frac{dx}{dt} + \frac{1}{\epsilon}\,\sin(\epsilon\,x) = 0, \tag{4.1}$$

where $0 < \epsilon \ll 1$. This system has a limit cycle: compute its approximate amplitude and period using the Poincaré-Lindstedt technique. Make sure that you compute at least the first nonvanishing correction to the linearized frequency. *Hint. There is a way to do this problem that requires a lot less algebra than a compute-blindly approach. Check the notes "Weakly Nonlinear Things: Oscillators", and use the fact that the argument of the sine in the equation is small.*

5 Multiple scales and limit cycles #03

Statement: Multiple scales and limit cycles #03

Consider the nonlinear oscillator described by the equation, for $\boldsymbol{x = x(t)}$,

$$\frac{\mathrm{d}^2x}{\mathrm{d}t^2} - \epsilon\,(1-x^4)\,\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \tag{5.1}$$

where $0 < \epsilon \ll 1$. This system has a limit cycle: compute its approximate amplitude and period using the method of multiple scales (also known as two-timing). The idea is to adapt the method presented in lectures for determining the amplitude of the limit cycle for the van der Pol oscillator.
Start by replacing the "single-time" dependence in $x = x(t)$ by a "two-times" dependence $\boldsymbol{x = x(t,\,\tau)}$, where $\boldsymbol{\tau = \epsilon\,t}$ is the "slow-time" over which the parameters in the harmonic oscillator ($\epsilon = 0$ case) solutions approximating the solutions to (5.1) evolve. Then

1. Then (from the chain rule) $\dfrac{\mathrm{d}x}{\mathrm{d}t} \mapsto \dfrac{\partial x}{\partial t} + \epsilon\,\dfrac{\partial x}{\partial \tau}$ and $\dfrac{\mathrm{d}^2x}{\mathrm{d}t^2} \mapsto \dfrac{\partial^2 x}{\partial t^2} + 2\,\epsilon\,\dfrac{\partial^2 x}{\partial t\,\partial \tau} + \epsilon^2\,\dfrac{\partial^2 x}{\partial \tau^2}$ in (5.1).
2. Use the expansion $x = x_0(t,\,\tau) + \epsilon\,x_1(t,\,\tau) + O(\epsilon^2)$, to **show that**
$$\big(\partial_{tt}\,x_0 + x_0\big) + \epsilon\big(\partial_{tt}\,x_1 + x_1 + 2\,\partial_{t\tau}\,x_0 + (x_0^4 - 1)\,\partial_t\,x_0\big) = O(\epsilon^2).$$
3. **Solve** the $O(1)$ system for x_0 real. Write your answer in terms of the complex amplitude $A(\tau)$.
4. Use your answer for x_0, and **compute the quantities** $2\,\partial_{t\tau}\,x_0$, x_0^4 and $\partial_t\,x_0$ in terms of A. By considering the secular terms that arise in the $O(\epsilon)$ system, **show that** $A(\tau)$ must satisfy
$$A_\tau = \frac{1}{2}\,A - A\,|A|^4. \tag{5.2}$$
5. Write A in polar form $A = r(\tau)\,e^{i\,\phi(\tau)}$, and **find the differential equations** satisfied by r and ϕ. **Determine the stable fixed point** of the radial equation.
6. **Deduce that** the limit cycle solution is (approximately) given by $\boldsymbol{x = 2^{3/4}\cos(t) + O(\epsilon)}$.

THE END.