Problem Set Number 05, (18.353/12.006/2.050)j
MIT (Fall 2024)

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Due: Thu. November 7, 2024 (turn it in via the canvas course website).

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1 Attracting Lines in the Phase Plane #01

Statement: Attracting Lines in the Phase Plane #01

You may have observed [1] the frequent appearance of special curves along which solutions tend to bunch up. Sometimes these are associated with the stable and/or unstable manifolds of certain critical points, and sometimes they are not. Below is a particular example, where you are expected to justify the special line with a scaling/asymptotic kind of analysis (coupled with an appropriate qualitative argument on the phase plane). The problem is aimed at testing that you can use these sort of tools properly, so do it using them. If you think of another way of doing it, and you want to show off, then I will look at it and consider it for extra credit — but this can backfire if this “other way” contains some grave conceptual error (thus do it only if you are certain).

Consider the damped pendulum equation:

$$\frac{d^2\theta}{dt^2} + a\,\frac{d\theta}{dt} + \sin\theta = 0, \qquad (1.1)$$

where $a \gg 1$. **Show that ALL the trajectories** in the phase plane $(\theta,\, \dot{\theta})$ end up, after a brief transient period, following the curve:

$$\frac{d\theta}{dt} \approx -\frac{1}{a}\,\sin\theta. \qquad (1.2)$$

Hint. Do a phase portrait with a computer to see what is happening.

Remark 1.1 (Warnings)

a. *Do not fall into the trap of just showing that $\dot{\theta} + \frac{1}{a}\sin\theta$ is small!* Both terms here are small, so showing this provides no new information. What you must show is that this expression is much smaller than either term, for example: $\dot{\theta} + \frac{1}{a}\sin\theta = O(a^{-2})$. In fact, it is possible to show that $\dot{\theta} + \frac{1}{a}\sin\theta = O(a^{-3})$.

b. *Do not fall into the trap of arguing that, just because some derivative is multiplied by a small parameter, you can neglect it* — this is not always true.[2] The qualitative argument in the phase plane is the step that will allow you to justify something like this.

c. *Show that **all** the trajectories (eventually) satisfy (1.2), not just that there are some that do.*

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[1] Say, in the phase plane plots with the problem set answers, or if you played with the MatLab scripts in the course toolkit.

[2] Examples where this fails are wide-spread in applications. They are not a mathematical curiosity.

2 Computer generated phase portrait: One Eye

Statement: Computer generated phase portrait: One Eye

A. Plot a computer generated phase plane portrait for the system

$$\dot{x} = y + y^2 \text{ and } \dot{y} = -x + \frac{1}{5}\,y - x\,y + \frac{6}{5}\,y^2, \tag{2.1}$$

in some "large" square, say: $-6 \leq x,\ y \leq 6$.
Note how many orbits eventually turn back towards the origin, to form the "eye".

B. Linearize the system near $(x,\ y) = (0,\ 0)$, and find **what type of critical point it is.**

C. Look at the phase portrait, and the result from **B**. **How are they consistent?**
Describe what happens near the critical point.

D. For $x \gg 1$, notice that the orbits approach a curve with $y \approx -1$. **Why does this happen?**

Hint for **C**. Do a detailed phase plane portrait near the eye.

Hint for **D**. The same type of approach used to analyze relaxation limit cycles works here, because something is large.

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3 Computer generated phase portrait: van der Pol #02

Statement: Computer generated phase portrait: van der Pol #02

Task #1. Plot a computer generated phase plane portrait for the

$$\textit{van der Pol oscillator:}\quad \ddot{x} - 2\,(1 - x^2)\,\dot{x} + x = 0. \tag{3.1}$$

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \dot{x}$, and plot in the square $-5 < u, v < 5$ [this will give you a nice plot including the "main features" in the phase portrait].

Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

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4 Counter-rotating limit cycles

Statement: Counter-rotating limit cycles

Provide answers to the queries below.

(1) **True or false:** Is it possible to have a (smooth) phase plane system with **exactly two periodic orbits,** one of which lies inside the other, such that: *the inner orbit runs counterclockwise, and the outer orbit runs clockwise.*
If true: do a sketch of a phase plane portrait with the stated property. **If false:** explain the reason.
Note that **the orbits would be limit cycles,** because they would have to be isolated.
Hint #1. Beware of "gut feeling" instinctive answers. Your intuition may be faulty!
Hint #2. There is no loss of generality in assuming that, if such a system exists, the orbits are concentric circles. Then it helps to think in terms of polar coordinates. See remark below.

(2) If the answer to the first query is "true", what is the minimum number of critical points required?

Remark. *How can you be sure that a system written in polar coordinates is smooth?* Answer: write the system in the form

$$\dot{r} = a\,r \quad \text{and} \quad \dot{\theta} = b, \tag{4.1}$$

where a and b are some functions of $(r,\ \theta)$ — equivalently, of $(x,\ y)$. In cartesian coordinates this corresponds to

$$\dot{x} = a\,x - b\,y = f, \quad \dot{y} = b\,x + a\,y = g. \tag{4.2}$$

Then $a = a(x,\ y)$ and $b = b(x,\ y)$ should be such that f and g are smooth. **For example, this happens if $a = a(r^2)$ and $b = b(r^2)$ are smooth functions of r^2.** However, it generally fails if they are functions of $r = \sqrt{x^2 + y^2}$ only, because r is not smooth at the origin, which would render (4.2) not smooth there. ♣

Question: *why do we transform (4.1) to cartesian coordinates in order to determine if the system is smooth?* Answer: to remove the coordinate singularity at the origin, which interferes with the task.

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5 Trapping regions; true or false

Statement: Trapping regions; true or false

True or false? There exists a phase plane system

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x,\, y) \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = g(x,\, y), \tag{5.1}$$

where f and g are fairly smooth (say, second partial derivatives continuous), and a simply connected (not empty) closed and bounded region $\mathcal{R}$, such that

A. $\mathcal{R}$ is a trapping region for the system in (5.1).

B. The system in (5.1) has no critical points in $\mathcal{R}$.

If true, give an example. If false, prove it. *Hint. Index theory and Poincaré Bendixon may help.*

An open set is simply connected if it has no holes. The technical definition is: *any loop (closed continuous curve) contained in the set can be continuously deformed into a point, while remaining inside the set.*

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6 Volume evolution

Statement: Volume evolution

Consider some arbitrary orbit, $\mathbf{\Gamma}$, for the system

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{F}(\vec{r}), \quad \text{where } \vec{r} \text{ and } \vec{F} \text{ are vectors in } \mathcal{R}^n, \tag{6.1}$$

and $\vec{F}$ has continuous partial derivatives up to (at least) second order. That is: $\mathbf{\Gamma}$ is the curve in $\mathcal{R}^n$ given by some solution $\boldsymbol{\vec{r} = \vec{r}_\gamma(t)}$ to (6.1). Then

A. Let $\Omega = \Omega(t)$ be an "infinitesimal" region being advected, along Γ, by the flow given by (6.1). *For example:*

A1. Let $\Omega(0)$ be a ball of "infinitesimal" radius $\mathrm{d}r$, centered at $\vec{r}_\gamma(0)$.

A2. For every point $\vec{r}_p^{\,0} \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (6.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^{\,0}$.

A3. At any time t_*, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^{\,0}$ runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a ball. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of hypervolume applies to it — see item **B**. In particular: *you do not need to use/know the formula for the hypervolume of a ball in n dimensions to do this problem!*

B. **Let $\boldsymbol{\mathcal{A} = \mathcal{A}(t)}$ be the hypervolume of $\boldsymbol{\Omega(t)}$.** Note: (i) if $n = 1$ the hypervolume is the length; (ii) if $n = 2$ the hypervolume is the area; (iii) if $n = 3$ the hypervolume is the volume; etc.

TASK #1. Find a differential equation for the time evolution of $\mathcal{A}$.

TASK #2. Optional. Use the differential equation that $\mathcal{A}$ satisfies to show that $\det(e^{B\,t}) = e^{\mathrm{tr}(B)\,t}$ for any square matrix B, where $\mathrm{tr}(B)$ denotes the trace of B. Note that you are required to do the proof using the differential equation, *specifically,* **not** by some other techniqe, like (say) linear algebra.

TASK #3. Optional. Prove the formula in (6.2) below.

Hints.

h1. Introduce the vector $\boldsymbol{\delta\vec{r}_p = \delta\vec{r}_p(t) = \vec{r}_p - \vec{r}_\gamma}$ for every point in $\Omega(t)$. This vector characterizes the evolution of the "shape" of Ω as the set moves along Γ. In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}_p$ vectors evolve.

h2. For every vector $\delta\vec{r}_p$, write an equation giving $\delta\vec{r}_p(t + \mathrm{d}t)$ in terms of $\delta\vec{r}_p(t)$ and the partial derivatives of $\vec{F}$ along Γ. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}_p(t)$ to $\delta\vec{r}_p(t + \mathrm{d}t)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(\mathrm{d}t)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t + \mathrm{d}t)$ and $\mathcal{A}(t)$. Note that:
(a) For linear transformations, hypervolumes are related by the absolute value of the determinant. †
(b) You need to calculate the determinant only up to $O(\mathrm{d}t)$ terms (neglect higher orders).
(c) For any square matrix M, $\boldsymbol{\det(1 + \epsilon\, M) = 1 + \epsilon\, \mathrm{tr}(M) + O(\epsilon^2)}$. (6.2)

h4. Item **h3** yields a formula of the form $\mathcal{A}(t + \mathrm{d}t) = \mathcal{A}(t) + (\text{something})\,\mathrm{d}t$.
Use this to get the differential equation for $\boldsymbol{\mathcal{A}}$.

† Multi-variable calculus: For a transformation $\vec{x} \to \vec{y}$; $\int f(\vec{y})\,\mathrm{d}\vec{y} = \int f(\vec{y}(\vec{x}))\,|\,\det(J)|\,\mathrm{d}\vec{x}$, where J = matrix of partial derivatives $\frac{\partial y_m}{\partial x_n}$. Thus for an infinitesimal hypervolume $\delta V \to |\det J|\,\delta V$. If $\vec{y} = M\,\vec{x}$ (linear transformation), $J = M$.

THE END.