Problem Set Number 04, (18.353/12.006/2.050)j MIT (Fall 2024)

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Due: Thu. October 24, 2024 (turn it in via the canvas course website).

Contents

1 Find a conserved quantity #03 (and sketch phase portrait)

Statement: Find a conserved quantity #03 (and sketch phase portrait)

Find a conserved quantity for the system and sketch the phase portrait. Include an analysis of any fixed point that occurs.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x + e^{-y} \quad \text{and} \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -y, \tag{1.1}$$

Hint: write the equation for the orbits, $\frac{\mathrm{d}x}{\mathrm{d}y} = g(x, y)$, and solve it.† *It will have a constant of integration, say β. This constant yields the conserved quantity, upon solving $x = X(y, \beta)$ for β as a function of x and y.*

† Note: write the equation for $\frac{\mathrm{d}x}{\mathrm{d}y}$, not $\frac{\mathrm{d}y}{\mathrm{d}x}$. The equation cannot be solved by separation of variables; however, upon multiplication by y you should be able to solve it.

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2 Hamiltonian for the Lotka-Volterra predator-prey model

Statement: Hamiltonian for the Lotka-Volterra predator-prey model

In non-dimensional variables, the Lotka-Volterra predator-prey model can be written in the form

$$\frac{dx}{dt} = x - x\,y = x\,(1-y) \qquad \text{and} \qquad \frac{dy}{dt} = -\mu\,y + \mu\,x\,y = \mu\,y\,(x-1), \tag{2.1}$$

where x is the prey (e.g.: mice), y is the predator (e.g.: owls), and $\mu > 0$ is a constant.

A. Discuss the biological meaning of each of the terms in the model.[1] That is: justify the model. In particular: (i) Comment on unrealistic assumptions. (ii) Explain the meaning of the non-dimensional constant μ.

B. Show that, in the first quadrant[2] ($x, y > 0$) the equations can be written in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \alpha\,H_y \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}t} = -\alpha\,H_x, \tag{2.2}$$

[1] Note that the model makes some rather drastic simplifying assumptions.

[2] The only region of biological interest.

for some functions $H = H(x, y)$ and $\alpha = \alpha(x, y) > 0$. Thus, upon an appropriate re-scaling of time along the solutions — given by $\boldsymbol{\alpha}\, \mathbf{d}\boldsymbol{t} = \mathbf{d}\boldsymbol{t}_{\mathsf{new}}$, the system becomes Hamiltonian.

C. Show that, in the first quadrant, $\boldsymbol{H}$ **is concave** $(H_{xx} + H_{yy} < 0)$, with a **unique strict global maximum at** $\boldsymbol{x = y = 1}$. Furthermore: $\boldsymbol{H \to -\infty}$ **along the edges of the quadrant, or as** $\boldsymbol{x^2 + y^2 \to \infty}$**. What does this imply about the solutions? Is this biologically realistic?**

Hint for part B. Since H must be a conserved quantity, the first step here is to find the conserved quantities. This you can do by looking at the equation $\frac{dy}{dx} = g(x, y)$, which can be integrated (you can get g from (2.1)). The integration involves a constant of integration — which thus is constant on each orbit. Hence, expressing this constant of integration as a function of x and y gives you a conserved quantity. Finally: note that if E is a conserved quantity, then so is $f(E)$ for any f with a non-zero derivative; thus you can re-cast an awkward looking E into a more convenient form.

Hint for part C. First show that H is concave in the quadrant, and that it goes to $-\infty$ at the edges and at infinity. It is a consequence of this that it has a unique strict global maximum.

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3 Nullclines versus stable manifolds

Statement: Nullclines versus stable manifolds

In the lectures we considered the example

$$\frac{dx}{dt} = x + e^{-y} \quad \text{and} \quad \frac{dy}{dt} = -y, \tag{3.1}$$

for which we drew its phase plane portrait.[3]

There is a somewhat confusing aspect of the phase portrait of this system: The nullcline $\dot{x} = 0$ has a similar shape and location as the stable manifold of the saddle, but they are not the same curve (which makes drawing the curves in the blackboard hard to do, even moderately accurate). **To clarify the relation between the two curves, plot both of them on the same phase portrait**, as follows:

1 Use a computer to do the plot, generating the stable manifold by numerically solving the equation.†

2 Find an explicit formula for the stable manifold, and then do a sketch of the phase portrait.†
Hint. Write the equation for $\frac{dy}{dx}$, and then solve it. A sketch without analytical justification is not acceptable.

† Note that the nullcline is known analytically.

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4 Reversible system that is not conservative

Statement: Reversible system that is not conservative

Give an example of a reversible system that is not conservative.

Hint. *Remember that a phase plane system*

$$\dot{x} = f(x, y) \quad \text{and} \quad \dot{y} = g(x, y), \tag{4.1}$$

is reversible if, for example, f is odd and g is even in y — that is: $f(x, -y) = -f(x, y)$ and $g(x, -y) = g(x, y)$. In this case the change $t \to -t$ and $y \to -y$ leaves the system invariant. Furthermore, we know that conservative systems cannot have sinks or sources. Now, ask yourself: what systems have exactly the opposite property [almost every critical point is either a source or a sink]. Then produce a system of this kind, with f odd and g even.

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[3] You can also find it in Matt Durey's Lectures_9_11_pre.pdf (bottom of page 1 and top of page 2), which is posted in the course web page. This is also Example 6.1.1 in Strogatz's book (First edition: pp. 147-148. Second edition: pp. 148-149).

5 Reversible system #03 (a "strange" reversible system)

Statement: Reversible system #03

Consider the system
where $\boldsymbol{E} = \boldsymbol{E}(x,\, y)$,
$a = a(x,\, y)$, and
$b = b(x,\, y)$, are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a\,E_y - b\,E_x \\ a\,E_x - b\,E_y \end{pmatrix} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix}\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \mathcal{A}\,\nabla E, \tag{5.1}$$

$$E = \frac{1}{2}\,y^2 - x^2 + \frac{1}{2}\,x^4, \quad a = 1, \quad b = \gamma\,x\,(1 - x^2)^2. \tag{5.2}$$

Here γ **is a positive constant.**

Now **do the following:**

- **a.** Show that $L = (-1,\, 0)$, $C = (0,\, 0)$, and $R = (1,\, 0)$, are **fixed points** — and *there are no others.*
  Hint. If you do all the algebraic operations that define the system, and then try to find the zeros, you will get a mess! Instead, notice that the fixed points are (exactly) the points where ∇E *vanishes. Explain why!*
- **b.** Show that **the system has a left-right ($x \mapsto -x$) time reversal symmetry.**
  Hint: E *and* a *are even in both* x *and* y*, while* b *is odd in* x *and even in* y*.*
- **c.** Show that **both L and R are linear centers,** while **C is a saddle.**
  When computing the Jacobian at the fixed points, it is useful to notice that both ∇E *and* b *vanish there.*
- **d.** Use a computer generated phase portrait to show that **both L and R are actually spirals!** — R is stable, while L is unstable. You should also check that **C is a saddle.** **Use $\gamma = 0.5$ for the portrait.**
- **e.** Recall now the theorem: *linear centers are true nonlinear centers for reversible systems.* **Why is it that items b and d do not contradict this theorem?**
- **f.** **Optional.** **Prove,** analytically, that: **R is a stable spiral** and **L is an unstable spiral.**
  Hint. Compute $\dot{E}$*, and notice that* L *and* R *are local minimums of* E*.*

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6 Systems both gradient and Hamiltonian 02

Statement: Systems both gradient and Hamiltonian 02

Consider a phase plane system which is both gradient and Hamiltonian:

$$\dot{x} = -V_x = \quad H_y, \tag{6.1}$$
$$\text{and} \quad \dot{y} = -V_y = -H_x, \tag{6.2}$$

for some potential $V = V(x,\, y)$ and Hamiltonian $H = H(x,\, y)$.

- **a. Show that both V and H satisfy the Laplace equation: $V_{xx} + V_{yy} = H_{xx} + H_{yy} = 0$.**
  You may assume that both V and H are twice continuously differentiable.
- **b.** Let $z = x + iy$ and consider the system
  $$\frac{\mathrm{d}z^*}{\mathrm{d}t} = i\,e^z, \tag{6.3}$$
  where * denotes the complex conjugate. Show that this system has the form in (6.1–6.2). What are the corresponding V and H?
  Recall that $e^z = e^x\,(\cos y + i\sin y)$.

THE END.