Problem Set Number 03, (18.353/12.006/2.050)j
MIT (Fall 2024)

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October 6, 2024

Due: Thu. October 17, 2024 (turn it in via the canvas course website).

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1 Attracting and Liapunov stable v2

Statement: Attracting and Liapunov stable v2

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} x(t) = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within δ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must approach* $\mathbf{x}^*$ *in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ stay within ϵ of $\mathbf{x}^*$ for all $t > 0$.
   In contrast with attracting, Liapunov stability requires nearby trajectories to remain close for all $t > 0$.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = y$ and $\dot{y} = -4x$.

b) $\dot{x} = 2y$ and $\dot{y} = x$.

c) $\dot{x} = 0$ and $\dot{y} = x$.

d) $\dot{x} = 0$ and $\dot{y} = -y$.

e) $\dot{x} = -x$ and $\dot{y} = -5y$.

f) $\dot{x} = x$ and $\dot{y} = y$.

2 Classify fixed points #02 (Linearize and find the fixed points type)

Statement: Classify fixed points #02

Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{y} - \boldsymbol{y^3},\ \dot{\boldsymbol{y}} = -\boldsymbol{x} - \boldsymbol{y^2}$. Then

a. Find the fixed points.

b. Linearize the equation near each fixed point, and classify the fixed points (saddles, stable nodes, etc.).

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3 Eliminate the cubic term

Statement: Eliminate the cubic term

Consider the system

$$\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4), \quad (3.1)$$

where $\boldsymbol{R \neq 0}$. The objective here is to find a new variable $\boldsymbol{x}$ such that the system transforms into

$$\frac{dx}{dt} = Rx - x^2 + O(x^4). \quad (3.2)$$

This is a big improvement: the cubic term is eliminated and the error term [1] is bumped to fourth order. [2] The procedure (sketched next) can be generalized to higher orders. [3]
Let $\boldsymbol{x = X + bX^3 + O(X^4)}$, where $\boldsymbol{b}$ is chosen later to eliminate the cubic term in the differential equation for x. This is called a **near-identity transformation**, since x and X are almost equal: they differ by a cubic term. [4] Now we need to rewrite the system in terms of x; this calculation requires a few steps.

1. Show that the near-identity transformation can be inverted to yield $\boldsymbol{X = x + cx^3 + O(x^4)}$, and solve for c.
2. Write $\boldsymbol{\dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4)}$, and substitute for X and $\dot{X}$ on the right hand side, so that everything depends only on x. Multiply the resulting series expansions and collect terms, to obtain $\boldsymbol{\dot{x} = Rx - x^2 + kx^3 + O(x^4)}$, where k depends on a, b, and R.
3. Choose $\boldsymbol{b}$ so that $\boldsymbol{k = 0}$.
4. **Explain where $R \neq 0$ is used.**

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4 Find and classify bifurcations problem #02

Statement: Find and classify bifurcations problem #02

This problem has three parts, and that in each you have to answer the same set of questions.
Part 1 of 3. For equation (4.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r\,x - \frac{x^3}{1 + 2\,x^2 + x^4}. \quad (4.1)$$

[1] Here the error is relative to approximating (3.1) by the saddle-node normal form $\boldsymbol{\dot{x} = Rx - x^2}$.
[2] Obviously we assume that both X and x are small.
[3] That is, one can successively eliminate all the higher order terms: $O(x^3)$, $O(x^4)$, ..., etc.
[4] We have skipped the quadratic term X^2, because it is not needed — you should check this later.

Check the PCS across $\boldsymbol{r=0}$ (see remark **4.1**). **Does it hold?**
Optional: Something "strange" happens for $\boldsymbol{r=0}$ in the bifurcation diagram. Is there another bifurcation taking place there? If so, which type? When you add it, does the PCS apply across $\boldsymbol{r=0}$? **Hint.** *Look at the equation satisfied by $y = 1/x$. What happens near $(y,\, r) = (0,\, 0)$?*

Remark 4.1 *The* **Principle of Conservation of Stability (PCS)** *says: Consider the ode $\dot{x} = f(x,\, r)$, where f is smooth and r is a parameter. Assign a weight $w = 1$ to each stable critical point, a weight $w = -1$ to each unstable critical point, and a weight $w = 0$ to each semi-stable critical point. Then the sum of the weights (* **stability index $\mathcal{S}$** *) is a constant (independent of $\boldsymbol{r}$).* ♣

Part 2 of 3. Consider the equation
$$\frac{dx}{dt} = r\,x - \frac{x}{\sqrt{1+x^2}}, \tag{4.2}$$
and repeat the analysis in part 1. **Important:** Be *careful* when doing the transformation to the variable y, as things are not entirely smooth at $y = 0$. It follows that what happens near $(y,\, r) = (0,\, 0)$ does not fit the "standard" canonical forms studied in the lectures. Nevertheless, you should be able to do it with minimum effort.

Part 3 of 3. Consider the equation
$$\frac{dx}{dt} = r\,x - x\,\mathrm{sech}(x), \tag{4.3}$$
and repeat the analysis in part 1. **Note:** the situation near $(y,\, r) = (0,\, 0)$ is even less "friendly" than the one in part 2. Yet, it is still tractable if you are careful.

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5 Ghosts and bottlenecks (Derive critical slowdown characteristic time by scaling)

Statement: Ghosts and bottlenecks

The aim of this question is to obtain an alternative derivation of the $T_{\text{bottleneck}} = O(r^{-1/2})$ scaling of the critical slowdown time for a system close to a saddle-node bifurcation, with $x(t)$ satisfying $\boldsymbol{\dot{x} = r + x^2}$ and $\boldsymbol{0 < r \ll 1}$. The idea is to do a variable re-scaling that reduces the ode to $\boldsymbol{\dot{u} = 1 + u^2}$. Since u does not depend on r, we can then "read" the bottleneck scaling from how time is transformed. *Proceed as follows:*

a. Suppose x has a characteristic scale $O(r^a)$, where a is unknown for now. Then $x = r^a u$, where $u \sim O(1)$. Similarly, suppose that $t = r^b \tau$, with $\tau \sim O(1)$.
Show that $\boldsymbol{u(\tau)}$ satisfies
$$\boldsymbol{r^{a-b}\frac{du}{d\tau} = r + r^{2a}u^2.} \tag{5.1}$$

b. Assuming that all terms in the equation have the same order with respect to r, derive the values of a and b. Deduce the bottleneck timescale.

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6 Linear System with Complex eigenvalues

Statement: Linear System with Complex eigenvalues

Note. Below λ complex means: λ has a non-vanishing imaginary part.
Here we consider, in some detail, the solution (and phase plane portrait) of a linear system where the eigenvalues are complex. The system we will look at is $\dot{x} = x - y$ and $\dot{y} = x + y$, which has the corresponding vector form $\dot{\mathbf{x}} = A\,\mathbf{x}$, where A is a 2×2 matrix.

a. Write A. Then show that it has the eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$, with corresponding eigenvectors $\mathbf{v}_1 = (i,\, 1)$ and $\mathbf{v}_2 = (-i,\, 1)$. **Notice: the eigenvalues, and corresponding eigenvectors, are complex conjugates. This is generic (always true) for a real 2×2 matrix A with complex eigenvectors.**#

\# **Proof.** Let A be a real square matrix, with λ a (complex) eigenvalue and corresponding eigenvalue $\mathbf{v}$; i.e.: $A\,\mathbf{v} = \lambda\,\mathbf{v}$. Taking the complex conjugate of this equation shows that $\bar{\lambda}$ is also an eigenvalue of A, with corresponding eigenvector $\bar{\mathbf{v}}$.

b. The general solution is then $\mathbf{x}(t) = c_1\, e^{\lambda_1\, t}\, \mathbf{v}_1 + c_2\, e^{\lambda_2\, t}\, \mathbf{v}_2$, for some constants c_1 and c_2. However, we are not yet done. This way of writing the solution involves complex numbers, and it does not make it clear what real valued solutions (which are the only ones we care about) do, nor what the orbits in the phase plane look like. Hence, **your task:** Express $\mathbf{x}$ purely in terms of real valued functions.† **Hint:** *use* $e^{i\,\omega\,t} = \cos\omega t + i\,\sin\omega t$ *to rewrite* $\mathbf{x}(t)$ *in terms of sines and cosines, and then collect the real and imaginary terms of the result.*†

† To get a real valued solution, the coefficients c_j in item **b** must be complex conjugates. **Hint:** use polar notation for the c_j.

Remark. The process described here can be applied to any 2×2 matrix with complex eigenvalues, using the fact that the real and imaginary part of the eigenvector(s) are linearly independent. The example here is specially simple because in this case the real and imaginary parts of $\mathbf{v}_1$ are the cartesian unit vectors.

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7 Numerical methods #01 (Test various numerical methods)

Statement: Numerical methods #01

Goal: test three numerical solutions, $x = x(t)$, for the initial value problem: $\dot{x} = -x$ with $x(0) = 1$.

a. Solve the problem analytically. What is the exact value of $x(1)$?

b. Use the Forward (or Explicit) Euler method to find the numerical approximation $\hat{x}_n(1)$ to $x(1)$ with a timestep of $\Delta t = 10^{-n}$ for $n = 0,\, 1,\, 2,\, 3,\, 4$. Let $E_n = |\hat{x}_n(1) - x(1)|$ be the error for each timestep. Plot $\log_{10}(\Delta t)$ versus $\log_{10}(E_n)$ and explain the results.

c. Repeat (b) using the Improved Euler method.

d. Repeat (b) using the fourth-order Runge-Kutta method.

The numerical methods above are displayed in Strogatz's book; in: §2.8 *Solving equations on the computer.*

THE END.