Problem Set Number 02, (18.353/12.006/2.050)j MIT (Fall 2024)

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1 Find and classify bifurcations problem #01

Statement: Find and classify bifurcations problem #01

For equation (1.1) below, find the values of r at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points x^* versus r.

$$\frac{dx}{dt} = r - \frac{x^2}{1+x^2}. \tag{1.1}$$

2 Get equation from phase line portrait problem #02

Statement: Get equation from phase line portrait problem #02

Consider the ode on the line

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{2.1}$$

where f is some function which has (at least) one continuous derivative. Assume that (2.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that x_1 is stable and that x_2 is unstable.[1] **Is this**

[1] A critical point is unstable if the solutions diverge from the critical point on both sides.

possible? Does a function $f = f(x)$ yielding this exist?
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

3 Get equation from phase line portrait problem #06

Statement: Get equation from phase line portrait problem #06

Consider the ode on the line

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x), \tag{3.1}$$

where f is some function which has (at least) one continuous derivative. Assume that (3.1) has exactly two critical points (i.e.: x_1 and x_2, with $-\infty < x_1 < x_2 < \infty$). Assume also that both critical points are unstable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

4 Phase line portrait problem #01

Statement: Phase line portrait problem #01

Consider the following ode on the line

$$\frac{dx}{dt} = f(x) = \text{sech}(x) - \frac{4}{5}. \tag{4.1}$$

Draw its phase line portrait, indicating the critical points, and their stability properties.

In addition: describe *quantitatively* the behavior of the solutions near the critical points (i.e.: at what rate do they approach or leave them), as well as the behavior of the solutions when $|x|$ is large. **In particular:** *Are there solutions that cease to exist for some finite value of t, or are the solutions valid for all times?*

5 The leaky bucket

Statement: The leaky bucket

The example here[3] shows that in some physical situations, non-uniqueness is natural and not pathological.

Consider a water bucket with a small hole in the bottom. If you see the bucket with a puddle beneath it, can you figure out when the bucket was full? Of course not! It could have finished emptying[4] 1 min ago, 10 min ago, ... The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Let us construct a simple model for the situation. Consider a cylindrical bucket, with constant cross-sectional area $\boldsymbol{A}$, and a small hole at the bottom, with area $\boldsymbol{a}$. The hole is small ($\boldsymbol{a} \ll \boldsymbol{A}$), so the water depth $\boldsymbol{h(t)} \geq \boldsymbol{0}$ (height of the liquid in the bucket at time $\boldsymbol{t}$) goes down slowly. Thus we assume that the bulk of the liquid is at rest, so

[2] A critical point is unstable if the solutions diverge from the critical point on both sides.
[3] Hubbard, J. H., and West, B. H. (1991) *Differential Equations: A Dynamical Systems Approach, Part I* (Springer, New York).
[4] Note that, in this problem, evaporation effects are neglected.

that the *rate at which the water goes down is controlled by the conversion of potential energy* in the bucket to *kinetic energy* in the exit water jet through the small hole (see remark **5.1**). For this purpose, let $\boldsymbol{v(t) \leq 0}$ be the water velocity through the hole (here we use the *convention* that $v > 0$ corresponds to water entering the bucket, while $v < 0$ if the water exits). Finally, we *neglect* surface tension and dissipation — the hole is not tiny.

[a] Show that $\boldsymbol{a\, v(t) = A\, \dot{h}}$. What physical law do you need to use here?

[b] To derive an additional equation, use conservation of energy. **First**, write an expression for the potential energy † in the bucket, $\boldsymbol{V}$, as a function of $\boldsymbol{h}$, $\boldsymbol{A}$, $\boldsymbol{g}$ (the acceleration of gravity), and $\boldsymbol{\rho}$ (the density of water). **Second**, write an equation for the rate at which kinetic energy is transported out of the bucket by the escaping water, $\boldsymbol{K_r}$, as a function of $\boldsymbol{v}$ and $\boldsymbol{m_r}$, where $\boldsymbol{m_r}$ is the rate at which mass leaves the bucket.‡

† Ignore the effect of air for this (no air pressure change over the depth of the bucket).

‡ You should be able to write $\boldsymbol{m_r}$ in terms of $\boldsymbol{\rho}$, $\boldsymbol{A}$, and $\boldsymbol{\dot{h}}$.

Finally, assume that all the potential energy is converted into kinetic energy, to obtain $\boldsymbol{v^2 = 2\, g\, h}$.

[c] Combine **[a]** and **[b]** to obtain the time evolution for $\boldsymbol{h}$ (in the form of a first order ode, with constant(s) written in terms of a, A, and g).

Important, be careful with the signs! Recall that $\boldsymbol{v,\ \dot{h} \leq 0}$. Further, it should be $\boldsymbol{m_r,\ K_r,\ V \geq 0}$.

[d] Given $\boldsymbol{h(0) = 0}$ (bucket empty at $\boldsymbol{t = 0}$), show that the solution for $\boldsymbol{h(t)}$ is **non-unique backwards in time**, i.e., for $\boldsymbol{t < 0}$. **Hint.** Find the solution to the ode, for all $\boldsymbol{t > t_*}$, given $\boldsymbol{h(t_*) = h_* > 0}$. Then use these solutions to produce solutions the backwards in time problem posed here.

[e] The description/derivation above ignores surface tension. Briefly discuss the effect surface tension, if significant (e.g.: tiny hole), would have on the outcome.

Remark 5.1 The problem is set-up in such a way that you do not need to know any fluid dynamics; you only need basic physics notions such as kinetic energy and potential energy. If you have familiarity with fluids, you may be tempted to, say, use Bernoulli's principle to derive the equation for $\boldsymbol{h}$; however: **do not do it! I want to see a derivation based on energy balance,** like the one outlined in **[a-c].** ♣

6 Toy model for column buckling

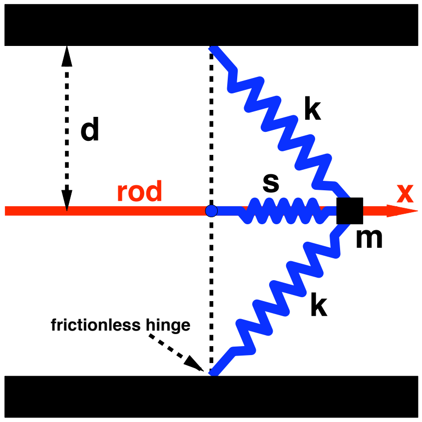
Statement: Toy model for column buckling

Imagine a vertical cylindrical elastic column, on which you push down along its axis by putting a weight on top. If the load is small enough, the column compresses a little, and the elastic response can balance the weight — with the cylinder staying straight. But, if the load is too large, this configuration is not stable, and the column buckles under the weight. This behavior arises because of the interplay of three forces: (i) the load; (ii) the elastic force along the axis of the cylinder; and (iii) the restoring force that is generated when the cylinder bends. When the axial forces are too large, the bending resistance is not enough to keep the straight state stable.

In this exercise we consider a very simple (1-D) toy model, exhibiting the essentials of the behavior described in the prior paragraph. Note though that it is an over-simplified "toy" model, where all the richness of the original setting is gone, and only the column buckling bifurcation remains.

A sketch depicting the model is shown in figure 6.1. Further assumptions and notation are:

1. The device is restricted to a plane, with the bead moving along a line.
2. Let $\boldsymbol{x}$ **be the distance,** along the rod, of the bead from the vertical line joining the spring supports. Let $x > 0$ if the bead is to the right of the supports and $x < 0$ if to the left. **Note the left-right ($x \to -x$) symmetry of the set-up**.



A bead of mass m (black square) can slide along a rigid horizontal rod (in red). The bead is connected by two equal springs (in blue), with spring constants k, to two supports placed symmetrically a distance d above and below the rod. A third spring, with spring constant s, pulls the bead towards the middle of the vertical line connecting the supports for the other springs. Everything is frictionless, except for the friction force opposing the motion of the bead along the rod. See the text for further details.

Figure 6.1: Toy model for column buckling.

3. The two main springs are equal, with a rest lengths $L > 0$ and spring constants $k > 0$. Each generates a force along its axis of magnitude (Hook's law) $F = k\,(\ell - L)$, where ℓ is the spring length. They push if $\ell < L$, and pull if $\ell > L$.

4. The spring aligned with the rod has zero rest length and a spring constant $s > 0$. This spring generates a restoring force $F = -s\,x$ along the rod, pulling the bead towards $x = 0$. *Note: a "better" model would have the restoring force provided by a torsion spring located at the hinge between the two springs on the bead. Such a spring would generate a restoring torque proportional to the angle between the two main springs.* However, there is no qualitative difference between these two set-ups — and the one here yields simpler algebra. [5]

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. The distance of the main spring supports from the rod is $\boldsymbol{d > 0}$. *Instead of considering the behavior of the system as a function of an applied compression force, we will consider it as a function of the total "imposed" length $2\,d$ of the "column".*

7. Because the rod is rigid, we need to consider only the horizontal components of the various forces that act on the bead. These are the forces provided by the three springs, and friction along the rod.

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.* [6]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the distance d changes (with everything else fixed). What type of bifurcation(s) occur?*

D. Consider the model that results from neglecting inertia. *The equation for this model can, with an appropriate scaling, be written in such a way that it contains a single a-dimensional parameter. Exhibit this form.*

To standardize the notation used in the answers, define $\displaystyle \boldsymbol{a = \frac{L}{d}}$ **and** $\displaystyle \boldsymbol{\gamma = \frac{s}{2\,k}}$**.** (6.1)

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[5] Both models are over-simplifications of the situation described in the first paragraph of the exercise. There is no point in worrying about getting small details right, when whooping simplifications occur elsewhere.

[6] Suggestion: to a-dimensionalize use d for length and $b/(2\,k)$ for time.

7 Taylor–von Neumann–Sedov blast wave radius

Statement: Taylor–von Neumann–Sedov blast wave radius

Consider what happens when a large amount of energy, $\boldsymbol{E}$, is released very fast (which we will approximate as "instantaneously") over a very small volume (which we will approximate as "at a point in space"). This in some quiescent gas. The result is the creation of a very strong spherical shock wave (the blast wave) propagating outwards from the energy release location, say $\vec{x} = \mathbf{0}$. We will also assume that the energy release occurs at time $\boldsymbol{t} = \mathbf{0}$.

The shock wave is very strong (at least initially) so that the pressure behind it is much larger than the pressure ahead (i.e.: $p_b \gg p_a$). Hence, for the analysis that follows we will neglect the pressure in the gas ahead of the shock (e.g.: p_a = atmospheric pressure).† On the other hand, we **cannot neglect the gas density ahead of the shock, $\rho_a > 0$.** This is because the density jump across a shock wave is known to be bounded; in other words: while $\rho_b > \rho_a$, it is not true that $\rho_b \gg \rho_a$.

† The argument is that p_a is so small compared to p_b, that we can safely make the approximation $p_a = 0$.

Task #1. What are the dimensions of $\boldsymbol{E}$ and ρ_a?

Task #2. Given the situation described above, use similarity analysis to produce a formula for the blast wave radius as a function of time, $\boldsymbol{R} = \boldsymbol{R}(\boldsymbol{t})$. This is a formula that involves $\boldsymbol{E}$ and ρ_a, where the only unknown is an a-dimensional multiplicative factor κ. Specifically: R is a length, and there is only one combination of E, ρ_a, and t, that yields a length.

Thus you should obtain a formula of the form

$$\boldsymbol{R} = \kappa\, \mathcal{A}(\boldsymbol{E},\, \rho_a,\, \boldsymbol{t}), \qquad (7.1)$$

where $\mathcal{A}$ is an algebraic expression with no free parameters.

Note. The gas velocity ahead of the blast wave vanishes, so it is not a parameter. On the other hand, the density ρ_b, pressure p_b, and velocity $\vec{u}_b$, of the gas behind can be written[7] once $R(t)$ is known, in terms of $p_a = 0$, ρ_a, and $\vec{u}_a = \mathbf{0}$. Hence p_b, ρ_b and $\vec{u}_b$, **are not** relevant to the similarity analysis that you are asked to do.

Task #3. Suppose that you know the value of κ, and that for some $t_1 > 0$, you are given $R_1 = R(t_1)$. Suppose also that you know ρ_a. Write now a formula for the energy $\boldsymbol{E}$ in terms of R_1, κ, ρ_a, and t_1.

7.0.1 Note on units and dimensions

You should not confuse dimensions with units. For example, a velocity has dimensions of length over time; **not** miles per hour, centimeters per second, leagues per day, ... Other examples: dimension(acceleration) = (length)/(time)^2, dimension(force) = (mass) × times (acceleration) = (mass) × (length)/(time)^2.

Things like miles, pounds, kilograms, meters, etc., are **units**. Units are specific quantities of a particular dimension, selected to measure the dimension. For standard units the selection is quite arbitrary, done for convenience, ease of use (at least for metric), and uniformity.† But in any particular setting, standard units may not be the most appropriate — e.g.: (i) Astronomers use the AU when dealing with solar systems, and light years (or parsecs) when dealing with stars; not meters or feet. (ii) Energy at the atomic level is measured in ev, not joules.

A key part of dimensional analysis is to pick the "right" units for a given problem; and there it must be that the units follow from the parameters that control the situation, *not arbitrary and unrelated things.*‡

† If everyone uses different units, it is hard to communicate, do commerce, or mass produce anything (in particular: tools).

‡ Like the length of the right foot of some king, which is how the foot used to be defined. Every kingdom then had a different foot, or some other part of the ruler's anatomy used to measure length: foot, pie, piede, punto, cana, palm, ...
And, of course, here at MIT we have the smoot.

Finally, **here is a simple example of similarity analysis.** Imagine that you have an object (which we idealize as a point) which, at time $t = 0$, is located at $\vec{x} = \mathbf{0}$ and not moving (zero velocity). Further, for $t > 0$ the object is subject to a constant acceleration $\vec{a}$. Find how the object's position evolves in time (without using calculus, nor even the definition of acceleration beyond its dimensions).

[7] Using the "Rankine-Hugoniot" conditions.

Answer. You need to construct a quantity with dimensions of length, $\vec{x}(t)$, using only an acceleration [with dimensions (length)/(time)^2] and time. Hence it must be $x = \kappa\,\vec{a}\,t^2$, **[A]**
where κ is a purely numerical constant. If this seems too simple, note that **[A]** is, basically,
what Galileo hypothesized (see **[g]**)

> *A falling body accelerates uniformly: it picks up equal amounts of speed in equaltime intervals, so that, if it falls from rest, it is moving twice as fast after two seconds as it was moving after one second, and moving three times as fast after three seconds as it was after one second.*

and then proved, using his famous *Acceleration Experiment*, where he rolled a ball down an inclined plane (to slow things down, and make measurement possible; he did not have high speed cameras).

Galileo also used a **scaling argument** (a cousin of similarity analysis) to argue that giants cannot exist. The idea is that muscle and bone strength scale (roughly) like their cross-section, while mass scales with the volume. Hence, beyond some size, muscles and bones are not be able to support the weight. *Corroborating evidence and related facts:* (i) elephant bones are much thicker relative their size than, say, rabbit bones. (ii) ants can carry loads much larger (relative to their weight) than, say, humans can. (iii) chitin exoskeletons only work for small creatures; for larger ones you need bones, which are stronger for a given cross-section. If titanium skeletons existed, there would be much larger land animals than elephants. (iv) On water the size limitation is higher, since the weight is then partially supported by the water. (v) How about dinosaurs? Well, a tyrannosaurus rex was roughly the same size as a large African elephant, and the larger dinosaurs seem to have been semi-aquatic — if any of you know more about this topic (dinosaurs), please correct me.

[g] Galileo Galilei Linceo, *Discorsi e Dimostrazioni Matematiche intorno à due nuoue scienze,* published in 1638 in Latin and Italian. For a translation into English see: http://files.libertyfund.org/files/753/0416_Bk.pdf *Dialogues concerning two new sciences, by Galileo Galilei.* Translated from the Italian and Latin by Henry Crew and Alfonso de Salvio. The Macmillan Company, NY, 1914.

THE END.