Problem Set Number 01, (18.353/12.006/2.050)j
MIT (Fall 2024)

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Statement: Computer exercises with a 1-D map v2.

The objective of this problem is to give an elementary first introduction to concepts such as: fixed point, stability, bifurcations, and chaos, via "experimental" (i.e., numerical) computation; plus a little bit of simple theory. We will do this by using a very simple mathematical model. This model is highly abstract, but we will argue later in the course that the model is also (to a significant degree) representative of the behavior of many real systems. In addition, this assignment also introduces the kind of computing and/or theoretical problems we shall often, but not always, assign.

In this model we will assume that the system is described by a single (scalar) time-dependent variable, $x(t)$. This variable could represent, say, the globally averaged temperature on the Earth's surface, the size of a particular population of animals on some secluded island (or in a Petri dish), some particular stock market average, etc. Furthermore, we will suppose that we are interested only in the values $x_n = x(t_n)$ at discrete times $t_n = n\,\Delta t$, where Δt is some suitable interval of time (say, a day). We then assume that the evolution of x in time may be written as

$$\boldsymbol{x_{n+1} = F(x_n)}. \tag{1.1}$$

where F is some function that describes the dynamics. For any of the examples mentioned above, it is obvious that the "true" F would involve complicated equations. Rather than going into that kind of detail, we will consider below two rather simple choices for F. That is:

$$\boldsymbol{F(x) = \mu\, x}, \tag{1.2}$$

and

$$\boldsymbol{F(x) = L\, p_f\, x\,(1 - x^2)}, \tag{1.3}$$

where $\boldsymbol{p_f = 1.5\sqrt{3}}$, while $\boldsymbol{\mu}$ and $\boldsymbol{L}$ are parameters (real constants) which vary in some range. **Below, after the remark and definition, are your tasks.**

Remark 1.1 *Notice that, for any function F: given x_n, all x_m with $m > n$ are uniquely determined — the forward "time" evolution is well defined.*†

† This simple observation is the key to the answer for some of the theory-tasks below.

However, unless F is invertible, the backward time evolution is **not** *well defined. There may be no x_{n-1} such that $x_n = F(x_{n-1})$, or there may be many. For example, let $F(x) = x^2$ and consider the cases $x_n < 0$ and $x_n > 0$.* ♣

Definition 1.1 Periodic orbits. *Given (1.1), and some x_0, the sequence $\{x_n\}$ is called an orbit for the dynamical system. An orbit is periodic, of period p ($p > 0$ an integer) if*

$$\boldsymbol{x_{n+p} = x_n},$$

for all n, and no integer $0 < q < p$ satisfies $x_{n+q} = x_n$.‡

‡ Obviously, if q is a multiple of p, $x_{n+q} = x_n$. Thus the period is the smallest positive integer satisfying $x_{n+q} = x_n$. Note that $p = 1$ corresponds to a *fixed point.* ♣

Note: In general, as $n \to \infty$, the orbits for (1.1) approach either a periodic or a chaotic # orbit, and periodic orbits are very important in the transition to chaos as some parameter in F varies — say, L in (1.3). For this reason below we pay particular attention to the periodic orbits. # See remark **1.2**.

Theoretical questions.

t1 Assume (1.1) and (1.2). Then, without using the computer, what can you predict about the behavior of x_n as $n \to \infty$, given an initial condition $x_0 \neq 0$?

t1a Qualitatively, what is different about the cases $|\mu| < 1$, $|\mu| > 1$, and $|\mu| = 1$?

t1b What is the qualitative difference between the evolution with $\mu > 0$ and $\mu < 0$?

t1c Check your conclusions numerically (do not hand in any plots for this, you can even use a calculator).

Note: In this case $x = 0$ is a *fixed point* of the evolution (i.e.: $x_n \equiv 0$ if $x_0 = 0$), and your conclusions, in particular, speak to the *stability* of this fixed point (what happens if a small perturbation is applied).

t2 Assume (1.1) and (1.3), with $0 \leq L \leq 1$. Show: F maps the unit interval onto itself — i.e.: $0 \leq F(x) \leq 1$ if $0 \leq x \leq 1$. Thus the unit interval is a suitable *phase space* for this dynamical system (phase space was defined in the lectures).

t3 Assume (1.1) and a generic F. Below: prove true or false.

t3a Assume $\{x_n\}$ is periodic of period p. Then $\{x_n\}$ cycles through p different values. Hint: see remark **1.1**.

t3b Assume $\{x_n\}$ cycles through *exactly* $p > 0$ different values, with every value taken at least twice. Then $\{x_n\}$ is periodic of period p.

Note: This fails for generic sequences; that this is a deterministic dynamical system matters. Examples: Flips of a coin with values head (1) or tail (0). Digits of the decimal representation of an irrational number.

Note: Twice each value matters. Example: $\{3,\, 1,\, -1,\, 1,\, -1,\, 1,\, -1\dots\}$, with $F(x) = -1 + 0.5\,(x-1)^2$, is not periodic if $x_0 = 3$ is included.

Hint: see remark **1.1**.

t3c It is possible to have a non-constant $\{x_n\}$ such that $x_{n+2} = x_n$ and $x_{n+5} = x_n$. Hint: see remark **1.1**.

t3d **Optional.** Let $\{x_n\}$ have period p, and assume $x_{n+q} = x_n$ for some $q > p$. Then q is a multiple of p.

Computer exploration. The aim is to numerically explore the system in (1.1), with F as in (1.3) and $0 \leq x,\ L \leq 1$ — see item **t2**. Specifically: **As a function of the parameter L, what is the behavior of the orbits $\{x_n\}$ as $n \to \infty$?** Further details (hints and tasks) follow.

c1 For this task a MatLab script, IterateCubiMap, is supplied. Please **read the script description at the top of the IterateCubiMap.m file**, which describes in detail what the script does and how to use it. IterateCubiMap was specifically designed to help with the required task: **How does x_n behave for n large?**

However, **you should not restrict yourself to this script only.** For example, from item **t3** it follows that you can get a lot of information (for a given L) by plotting/marking in the interval $0 \leq x \leq 1$ the points visited by the sequence $\{x_n\}$ for n large (this so the sequence has time to achieve its asymptotic behavior). Then:

c1a If the orbit visits only a finite set of points, then it is periodic, and the period is the number of points.

c1b Chaotic orbits will fill regions, containing an infinite number of points (but see remark **1.2**).

Note. For now we will not attempt to define chaos. Just "looks random" is enough — see remark **1.2**.

Note. Of course, with a computer you cannot plot "an infinite number of points"; but you can detect orbits that visit a very large number of points.

Note. Visiting an infinite number of points is not enough to characterize a discrete dynamical system orbit as chaotic; but for us here this will do. Further, the regions visited by a chaotic orbit are not "intervals"; in fact they can be quite complicated (fractals); though it would be very hard for you to detect their structure with the tools you have.

You should be able to write your own program implementing the idea in this paragraph (not required, though).

c2 The system (1.1/1.3) exhibits many behaviors. For example, x_n may approach a constant (a fixed point) as $n \to \infty$, or it may approach a *periodic cycle*, where $x_{n+p} = x_n$ (p is the period), or the $n \to \infty$ behavior may be chaotic (see remark **1.2**), or it may exhibit *intermittent chaos*, where the sequence alternates (seemingly randomly) between being close to a periodic cycle, and chaotic bursts. [1] These various behaviors correspond to different values of $\boldsymbol{L}$, with the transition from one to another occurring at *critical values* of $\boldsymbol{L}$ (of which there are many; infinitely many, in fact).

A value $\boldsymbol{L = L_c}$ is called a critical value if the long term evolution for $\boldsymbol{x_n}$ changes *qualitatively* as $\boldsymbol{L}$ crosses $\boldsymbol{L_c}$. These qualitative changes are called **bifurcations**. For example, on one side of $\boldsymbol{L_c}$ the system may be attracted to some fixed point $\boldsymbol{x_*}$, and to a different fixed point on the other side. Or maybe the behavior switches from fixed point to periodic of some period $\boldsymbol{p > 1}$, or the period changes across $\boldsymbol{L_c}$, etc.

These are the questions you are asked to investigate:

c2a There is a critical value, $\boldsymbol{L_1}$, such that for $L < L_1$ the orbits converge to $\boldsymbol{x = 0}$, and for $\boldsymbol{L_1 < L < L_2}$ they converge to a constant $\boldsymbol{x = x_* > 0}$. Find L_1 and L_2 (approximately) — *it is very hard to accurately compute the critical values numerically, and you are not being asked to do so.*
Optional. In fact, L_1 and x_* are easy to compute analytically. *Can you do so?*
Hint: Linearize the map for $\boldsymbol{x}$ small, and use item **t1**.
Note that $\boldsymbol{L_1}$ and $\boldsymbol{L_2}$ are *critical values, or bifurcation points.*

c2b What happens for $\boldsymbol{L}$ slightly above $\boldsymbol{L_2}$?

c2c In fact, there is an infinite sequence of critical points $\boldsymbol{L_1 < L_2 < L_3 < \ldots}$, with $\boldsymbol{\lim_{n\to\infty} L_n = L_\infty}$ (where $\boldsymbol{0.88 < L_\infty < 0.89}$). In each "window" $\boldsymbol{L_n < L < L_{n+1}}$, the limit behavior of $\boldsymbol{x_n}$ is very simple — in part **c2b** you should already have discovered what happens for $\boldsymbol{L_2 < L < L_3}$. **Describe the behavior in these windows.** Note: it is very hard to compute in any detail anything beyond L_4,‡ but you can tell what happens for $\boldsymbol{L_4 < L < L_5}$ without knowing L_5. **From these first few windows you should be able to figure out (guess) what the pattern is for $\boldsymbol{L_n < L < L_{n+1}}$.**

‡ The window width $L_{n+1} - L_n$ decreases exponentially with n. Very quickly you will need more digits than you have to tell the L_n them apart.

c2d What happens slightly beyond $\boldsymbol{L_\infty}$. Do your best here; see remark **1.2**.

c2e **What do you see beyond $\boldsymbol{L_\infty}$, as you move up towards $\boldsymbol{L = 1}$.** Let us see how much can you find (there is infinite detail here, so the sky is the limit). Hint. More critical values, more periodic windows, more chaos, intermittent chaos.

Remark 1.2 *For this problem we will not define chaos (will be done later). Here simply check that the behavior is not periodic of any "reasonable" period,*† *e.g.:* $\boldsymbol{1 \leq p \leq 16}$ *(many can be excluded by simple eye-sight).* ♣

† Because you are doing this in a computer, there is only a finite number of values $\boldsymbol{x_n}$ can take, so the computed sequence $\boldsymbol{x_n}$ will always be periodic ... but the period can be huge, for any practical purposes "infinite".

THE END.

[1] The MatLab script provided allows you to check for these behaviors.