Problem Set Number 08, (18.353/12.006/2.050)j
MIT (Fall 2023)

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Due day of last lecture, Fall 2023 (turn it in via the canvas course website).

Contents

1 Generalized Cantor sets

Statement: Generalized Cantor sets

Suppose that we construct a new kind of Cantor set, by removing the middle **half** of each subinterval, rather than the middle third.

- **a.** Show that the *length* of the resulting set still vanishes, same as for the regular Cantor set.
- **b.** Find the similarity dimension of the set.
- **c.** Generalize the construction so as to produce a Cantor set with zero length and with a similarity dimension that can be picked as any arbitrary number in $0 < d < 1$.

2 Nonlinear stability of a discrete map, and flip bifurcation

Statement: Nonlinear stability of a discrete map, and flip bifurcation

Consider a 1-D map, $x_{n+1} = f(x_n)$, where f is smooth. Assume a fixed point $x_f = f(x_f)$, where $f'(x_f) = -1$ — hence linearization does not determine the stability of x_f. Without loss of generality, assume $x_* = 0$, and write

$$f(x) = -x + a\,x^2 + b\,x^3 + O(x^4), \tag{2.1}$$

where a and b are constants. These are **your tasks:**

- **t1.** **Find condition on a and b** that determines wether $x = 0$ is a stable or unstable fixed point. *Hint:*
  - t1.a The condition looks like: stability if $h(a,\, b) > 0$, and instability if $h(a,\, b) < 0$, for some function h.
  - t1.b Consider what happens upon iterating $g(x) = f(f(x))$, which you can ascertain by expanding g to $O(x^4)$, using (2.1). Then note: if $x_{2\,n+2} = g(x_{2\,n})$ decays/grows, then so does $x_{2\,n+3}$, because f is continuous.
- **t2.** **Answer this question:** *why do you have to expand g up to $O(x^4)$, in item* t1.b*, to determine stability?* Note that here expect the mathematical/technical reason for this.
- **t3.** Let a and b in (2.1) be such that $x = 0$ is stable, i.e.: $h(a,\, b) > 0$, and take a map F such that

$$F(x) = -(1+\delta)\,x + a\,x^2 + b\,x^3 + O(x^4), \tag{2.2}$$

  where $0 < \delta \ll 1$. Then x is a linearly unstable fixed point, and a **period two (stable) solution**

$$x^*_{n+2} = x^*_n, \quad x^*_{n+1} = F(x^*_n), \tag{2.3}$$

  appears,[‡] where x^*_n has size $O(\sqrt{\delta})$.

This is called a **supercritical (or soft) flip bifurcation.**

‡ Argument: the same we made to explain the scaling behind supercritical pitchfork and Hopf bifurcations.

The new solution appears as a balance between the destabilizing linearity, and the stabilizing nonlinearity.

Your task. Pick an example F where this happens, with $a \neq 0 \neq b$, and show a numerically computed picture of cobwebs† converging to the period two stable solution.

† Use two cobwebs (with different colors), one converging from "inside" and the other from "outside".

I suggest that you write a "generic" program for $F(x) = -(1+\delta)\,x + a\,x^2 + b\,x^3$ *and initial data* x_0*, and then play with the parameters till you get a pretty picture. Further: choose your colors well; e.g.: yellow on a white background is a bad idea! Note: something like* $1 < a < 2$*,* $b \sim -2/3$*, and* $\delta \sim 0.3^2$*, worked for me.*

3 Sierpinski gasket

Statement: Sierpinski gasket

Consider the fractal (a "Sierpinski gasket") in the plane, made in the following **recursive fashion:**

1. Start with an equilateral triangle, with sides of length L.
2. Draw the lines joining the sides mid-points, and divide it into four equal equilateral sub-triangles.
3. Remove the sub-triangle at the center.
4. Repeat the process with each of the other three remaining sub-triangles.

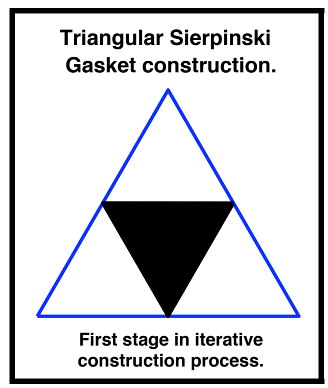

Figure 3.1: The picture on the right illustrates the recursion, showing the result of the first iteration in the process described above.

Now, do the following:

A. Calculate the box dimension of the fractal.

B. Calculate the self-similar dimension of the fractal.

C. Calculate the surface area of the fractal.

D. **Optional.** Show that the fractal has as many points as a full square — **This part is hard(er).**

E. **Optional.** Let d_s be the dimension calculated in part **A**. Modify the construction of the fractal, in such a way that the modified fractal can be selected to have **any** given box dimension $0 < d < d_s$.
Hint: take out bigger chunks at each stage.

F. **Optional.** Construct fractals (subsets of the plane) such that their box dimensions can be selected to have **any** given box dimension $d_s < d < 2$.

THE END.