

Problem Set Number 06, (18.353/12.006/2.050)j

MIT (Fall 2023)

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1 Homoclinic bifurcation with a computer #01

Statement: Homoclinic bifurcation with a computer #01

Consider the system $\dot{x} = \mu x + y - (2\mu + 1)x^2$ and $\dot{y} = -x + \mu y + (2 - \mu)x^2$, (1.1) where μ is a parameter. For this system:

(1) Find and analyze/classify the critical points.

(2) Using a computer, show that a supercritical Hopf bifurcation happens at $\mu = 0$.

(3) Using a computer, show that an homoclinic bifurcation happens for $0 < \mu < \mu_c$, where μ_c is very small.

You can get a rough idea of “how small is small” by recalling that the “radius” of the Hopf bifurcation limit cycle grows like the square root of the parameter deviation from the onset. Hence, in this case it should be $C\sqrt{\mu_c} \approx$ distance from the origin to the saddle, where C is a constant. Of course, you do not know C , so this estimate can be off by some factor, probably less than 10. Do this estimate before doing (3).

(4) Find the approximate value of μ_c .

(5) Illustrate your results with phase plane plots in the region $-0.4 < x, y < 0.6$.

Hint: The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence check the behavior of the orbits near the critical point for $\mu = 0$, to see if the nonlinear terms stabilize or de-stabilize.

Suggestions for computation. Use PHPLdemoB2, which allows for easy parameter searches. To get precisely controlled plots, you can use PHPLplot or PHPLplot_v2, which are a bit less user friendly, but allow precise control. Note that, for μ small, the radial dynamics near the origin is slow, thus you may need “large” integration times ... do not use the default integration times and tolerances in the scripts.

2 Hopf bifurcation using a computer #01

Statement: Hopf bifurcation using a computer #01

For the system $\dot{x} = y + \mu x$ and $\dot{y} = -x + \mu y - x^2 y$, (2.1) a Hopf bifurcation occurs at the origin when $\mu = 0$.

Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical.

For small values of μ , verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.

Hints: (a) The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence check the behavior of the orbits near the critical point for a $|\mu| \ll 1$ to see if the nonlinear terms stabilize or de-stabilize. (b) In the subcritical case the limit cycle is unstable, hence hard to compute forward in time. However, if you run the system backwards in time the solutions converge to the limit cycle!

Suggestions for computation. For the calculation of the limit cycles, and measuring the radius and period, PHPLdemoB is not the best tool (because it depends on the mouse to input the initial point on an orbit. For this you should use [PHPLplot](#) or [PHPLplot_v2](#), which are a bit less user friendly, but allow precise control of the orbits. With them you can select the initial data (say, give $x = x_0 > 0$ along $y = 0$) and play with x_0 till you hit the limit cycle — set $\text{BF} = 0$, so the orbit is computed forward only. Then x_0 gives you the “radius”. To compute the period, play with $t.f$ so the orbit barely closes — alternatively: “hack” the script so it outputs the raw data from the ode solver, $[T, Y]$, which will allow you to compute the period and radius very accurately.

3 Limit cycle bifurcation with a computer #01

Statement: Limit cycle bifurcation with a computer #01

The system $\dot{x} = y/\delta$ and $\dot{y} = (2\mu y - (1 + \mu^2)x + x^2(\mu x - y))/\delta$, (3.1)

where $\delta = 1 + x^2$, undergoes a

supercritical Hopf bifurcation at $\mu = 0$. Beyond that, as μ increases, while the critical points at $(x, y) = (\pm x_0, 0)$ ($x_0 = \sqrt{\mu + 1/\mu}$) move towards the origin. For $\mu = \mu_c$, $0.3 < \mu_c < 0.4$, another bifurcation occurs. Using a computer:

- (1) Describe what happens at $\mu = \mu_c$.
- (2) Find the nature of all the critical points of the system.
- (3) Ascertain the value of μ_c with two significant digits.

Illustrate your conclusions with three phase plane portraits in the region $-3 < x, y < 3$, one for $\mu < \mu_c$, one for $\mu > \mu_c$, and one for $\mu \approx \mu_c$.

Remark #1. Note that the system has the symmetry $(x, y) \rightarrow (-x, -y)$.

Suggestions for computation. Since the problem involves investigating a situation as a parameter varies, with an equation with many terms, PHPLdemoB is not the best tool (impractical). Instead, I suggest that you use [PHPLdemoB2](#), which will allow you to do μ “sweeps” efficiently. Once you know well what happens, to produce “neat” final plots, you could use [PHPLplot](#) or [PHPLplot_v2](#), which are less user friendly than PHPLdemoB2, but allow precise control of the orbits.

4 Orbit diagrams for 1-D maps #01

Statement: Orbit diagrams for 1-D maps #01

Compute the orbit diagrams¹ for the 1-D maps $x_{n+1} = f(x_n)$ below. Furthermore, in all the cases perform the following **task:** given the range of r selected, justify the selected range for x .

Meaning of “justify”. Show that the x -region is such that it may contain an attractor, because either: (a) It is trapping; an orbit starting there stays there; or (b) Orbits starting outside the region diverge to infinity, so that any attractor has to be inside.

1. The **sine map** $f(x) = r \sin(x)$, with $-5 \leq r \leq 5$ and $-5 \leq x \leq 5$.

¹ An orbit diagram is what figure 10.2.7 in Strogatz’s book shows.

2. The **cubic map** $f(x) = rx - x^3$, with $0 \leq r \leq 3$ and $-2 \leq x \leq 2$.
3. The **exponential #1 map** $f(x) = xe^{r(1-x)}$, with $1 \leq r \leq 4$ and $0 \leq x \leq 0.25e^3 = 5.0213\dots$
4. The **tent map** $f(x) = r(1 - |2x - 1|)$, with $0 \leq r \leq 1$ and $0 \leq x \leq 1$. At first sight the tent map orbit diagram may look rather plain. However, *do a blow up of the region in the tent map orbit diagram given by:*
 - 4b.** $0.5 \leq r \leq 0.6$ and $0.492 \leq x \leq 0.501$.
 - 4c.** $0.5 \leq r \leq 0.525$ and $0.4999 \leq x \leq 0.5001$.

Note that doing these blow-ups will require to plot far more iterates than for the main plot, because you will not be plotting the whole attractor, but just a small region within it, which means most points will be outside the plotted region.

Process: Pick a grid in r ; say $r_m = r_0 + m\Delta r$, for some small Δr , covering the desired range for r . Then, for every r_m , take a random starting point x_0 , and calculate the resulting x_n for a large number of iterates $1 \leq n \leq N$. Finally: throw away the first few hundred iterates (to eliminate transients as the attractor is approached) and plot the rest as points (r_m, x_n) in a 2-D diagram.

Note: if you use MatLab, “vectorize” the operation, so that you do **all** the r simultaneously (MatLab will be very slow if you use a “for loop” in both n and m).

Also, in MatLab the command “print -dpng FigureName” will save the figure as a small png file (and it is more reliable than trying to use the GUI in the figure window).

THE END.