Problem Set Number 05, (18.353/12.006/2.050)j
MIT (Fall 2023)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

November 3, 2023

Due Tue. November 14, 2023 (turn it in via the canvas course website).

Contents

1 Computer generated phase portrait: The Eyes

Statement: Computer generated phase portrait: The Eyes

First, **plot the phase plane portrait** for (the eyes system) using a computer. I encourage

$$\dot{x} = y + y^2 \quad \text{and} \quad \dot{y} = -\frac{1}{2}\,x + \frac{1}{5}\,y - x\,y + \frac{6}{5}\,y^2, \tag{1.1}$$

you to *use the* PHPLdemoB *MatLab script* provided with the class Toolkit. **Do two plots, say: $-7 < x,\ y < 7$ ("large scale") and ($-6 < x < 4$; $-4 < y < 3$) for a bit more detail.**

Next, **find the critical points and classify them — does what you observe in the plot match what the theory predicts? Explain any discrepancies.** *Be careful with this part!* Explore (numerically) what happens close to the critical points; *what do you see there and how does it make the portrait consistent with the theory?*

Optional tasks.

(1) Look at the phase portrait. There is an "obvious" visual symmetry that you should spot. **Explicitly write the symmetry of the system** that explains what you see (a change of variables that leaves the system invariant).

Hint. Look at the critical points. What transformation (consistent with the visual symmetry) maps one to the other?

(2) In your "large scale" portrait you should be able to see that the orbits, for $|x|$ large, approach a curve with $y \approx -0.5$ (for $x > 0$) or leave it for $x < 0$. **Make a theoretical argument explaining why this happens.**

Hint. The same type of approach used to analyze relaxation limit cycles works here, because something is large.

2 Computer generated phase portrait: van der Pol #01

Statement: Computer generated phase portrait: van der Pol #01

Task #1. Plot a computer generated phase plane portrait for the *van der Pol oscillator:*

$$\ddot{x} - 2\,(1 - x^2)\,\dot{x} + 4\,x = 0. \tag{2.1}$$

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \frac{1}{2}\,\dot{x}$, and plot in the square $-5 < u, v < 5$ [this will give you a nice plot including the "main features" in the phase portrait].

Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

---

3 Computer generated phase portrait: van der Pol #03

Statement: Computer generated phase portrait: van der Pol #03

Task #1. Plot a computer generated phase plane portrait for the *van der Pol oscillator:*

$$\ddot{x} - 4\,(1 - x^2)\,\dot{x} + x = 0. \tag{3.1}$$

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \dot{x}$, and plot in the square $-6.5 < u, v < 6.5$ [this will give you a nice plot including the "main features" in the phase portrait].

Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

---

4 Find the potential for a gradient system

Statement: Find the potential for a gradient system

Suppose that you are given a system $\dot{x} = f(x,\, y)$ and $\dot{x} = g(x,\, y)$. (4.1)
How can you to tell if (4.1) is a gradient system? Answer: **A necessary condition is $f_y = g_x$.** (4.2)
Why? Because if (4.1) is a gradient system, for some potential
$V = V(x,\, y)$, $f = -V_x$ and $g = -V_y$.
In N dimensions, $\dot{\vec{x}} = \vec{f}(\vec{x})$, the condition (4.2) becomes $(f_n)_{x_m} = (f_m)_{x_n}$ for all n and m.
It turns out that, if the system is defined **in a simply connected region, (4.2) is also sufficient.** (4.3)
Then we can write $V(\vec{r}) = -\int_\Gamma (f\,\mathrm{d}x + g\,\mathrm{d}y)$, (4.4)
where Γ is a curve from some fixed point $\vec{p}$ in the region, to $\vec{r} = (x,\, y)$.
Because of the Green-Stokes theorem, the integral in (4.4) does not depend on the choice of Γ. *Make sure that you understand why this is so!*

If the region in (4.3) is a rectangle with sides parallel to the axes, this simple process works: (4.5)
First, integrate $V_x = -f$ for each value of y. This yields V up to some unknown, additive, function $h = h(y)$ (the integration constant). Then find h by using the other equation $V_y = -g$; which determines h up to a constant. *The reason that this works* is because a V exists, as shown by (4.4).

Your task: for the systems below, determine which ones are gradient systems and which ones are not, and find V for the ones that are gradient. The systems are defined everywhere in the plane.

(a) $\dot{x} = 2\,x\,y - y\,\cosh(x)$ and $\dot{y} = x^2 - \sinh(x)$.

(b) $\dot{x} = 3\,x^2 - e^y\,\cos(x)$ and $\dot{y} = -1 - e^y\,\sin(x)$.

(c) $\dot{x} = 3\,x^2 - e^y\,\sin(x)$ and $\dot{y} = -1 - e^y\,\cos(x)$.

---

5 Planetary orbits in General Relativity

Statement: Planetary orbits in General Relativity

The relativistic equation for the orbit of a planet around a star is
$$\frac{\mathrm{d}^2 u}{\mathrm{d}\,\theta^2} + u = \alpha + \epsilon\, u^2, \qquad (5.1)$$
where $(r,\, \theta)$ are the polar coordinates for the planet's position in the plane of motion, and $u = 1/r$. The parameter $\alpha > 0$ is related to the angular momentum of the orbit (it is the same as in classical Newtonian mechanics). Finally, the term ϵu^2 is the relativistic correction to Newtonian mechanics, where $0 < \epsilon \ll 1$. Note: we are **only interested in solutions with $u \geq 0$**. Note that $u = 0$ somewhere corresponds to the planet escaping the star's gravitational field.

(a) Rewrite the equation as a system in the $(u,\, v)$ plane, where $v = \dfrac{\mathrm{d}u}{\mathrm{d}\theta}$.

(b) Find all the equilibrium points of the system.

(c) Show that one of the equilibria is a center in the $(u,\, v)$ phase plane, according to the linearization. *Is it a nonlinear center?*

(d) Show that the equilibrium point found in **(c)** corresponds to a circular planetary orbit.

(e) **Optional.** The equation has solutions where u is a periodic function of θ.
e1. *Do these solutions correspond to periodic orbits around the star?*
e2. *If not, what do they correspond to?*
e3. *What happens when $\epsilon = 0$ (Newtonian mechanics)?*
Hint. Examine first **e3.** *Where are the elliptical orbits?*

---

6 Saddle connections

Statement: Saddle connections

Consider a phase plane system with exactly two fixed points, both of which are saddles. Consider now the following situations:

(a) There is no orbit that connects the saddles.

(b) There is exactly one orbit that connects the saddles.

(b) There are exactly two orbits that connect the saddles.

In each case, **if possible, sketch a phase portrait for such a system, else give a reason for why it is not possible. The sketches MUST include what the stable and unstable manifolds of the two saddles do.**

7 Two nested limit cycles and a single critical point

Statement: Two nested limit cycles and a single critical point

(1) Give an example of a (smooth) phase plane system with exactly two periodic orbits and a single critical point. Note: the two periodic orbits will necessarily by isolated, hence they will be limit cycles. *Hint: easy to do in polar coordinates (see remark below).*

(2) **True or false:** In the situation of item **1**, both limit cycles can be repellers (i.e.: trajectories near the limit cycles diverge from them). **If true**, sketch an appropriate phase plane portrait. **If false:** explain why. *Hint: recall the Poincaré Bendixon theorem.*

Remark. *How can you be sure that a system written in polar coordinates is smooth?* Answer: write the system in the form

$$\dot{r} = a\,r \quad \text{and} \quad \dot{\theta} = b, \tag{7.1}$$

where a and b are some functions of $(r,\ \theta)$ — equivalently, of: $(x,\ y)$. In cartesian coordinates this corresponds to

$$\dot{x} = a\,x - b\,y = f, \quad \dot{y} = b\,x + a\,y = g. \tag{7.2}$$

Then $a = a(x,\ y)$ and $b = b(x,\ y)$ should be such that f and g are smooth. **For example, this happens if $a = a(r^2)$ and $b = b(r^2)$ are smooth functions of r^2.** However, it generally fails if they are functions of $r = \sqrt{x^2 + y^2}$ only, because r is not smooth at the origin, which would render (7.2) not smooth there. ♣

Question: *why do we transform (7.1) to cartesian coordinates in order to determine if the system is smooth?* Answer: to remove the coordinate singularity at the origin, which interferes with the task.

THE END.