Problem Set Number 04, (18.353/12.006/2.050)j
MIT (Fall 2023)

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Contents

1 Volume evolution

Statement: Volume evolution

Consider some arbitrary orbit, $\mathbf{\Gamma}$, for the system

$$\frac{\mathrm{d}\vec{r}}{\mathrm{d}t} = \vec{F}(\vec{r}), \quad \text{where } \vec{r} \text{ and } \vec{F} \text{ are vectors in } \mathcal{R}^n, \tag{1.1}$$

and $\vec{F}$ has continuous partial derivatives up to (at least) second order. That is: $\mathbf{\Gamma}$ is the curve in $\mathcal{R}^n$ given by some solution $\boldsymbol{\vec{r} = \vec{r}_\gamma(t)}$ to (1.1). Then

A. Let $\Omega = \Omega(t)$ be an "infinitesimal" region that is being advected, along Γ, by the flow given by (1.1). *For example:*

- **A1.** Let $\Omega(0)$ be a ball of "infinitesimal" radius $\mathrm{d}r$, centered at $\vec{r}_\gamma(0)$.
- **A2.** For every point $\vec{r}_p^{\,0} \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (1.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^{\,0}$.
- **A3.** At any time t_*, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^{\,0}$ runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a ball. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of hypervolume applies to it — see item **B**. In particular: *you do not need to use/know the formula for the hypervolume of a ball in n dimensions to do this problem!*

B. Let $\boldsymbol{\mathcal{A} = \mathcal{A}(t)}$ be the hypervolume of $\boldsymbol{\Omega(t)}$. Note: (i) if $n = 1$ the hypervolume is the length; (ii) if $n = 2$ the hypervolume is the area; (iii) if $n = 3$ the hypervolume is the volume; etc.

TASK: Find a differential equation for the time evolution of $\mathcal{A}$.

Optional: use the differential equation to show that $\det(e^{B\,t}) = e^{\mathrm{tr}(B)\,t}$ for any square matrix B — where $\mathrm{tr}(B)$ denotes the trace of B. Note: what you are asked to do here is to do the proof using the differential equation, *specifically,* **not** by some other techniqe, like (say) linear algebra.

Hints.

h1. Introduce the vector $\boldsymbol{\delta\vec{r}_p = \delta\vec{r}_p(t) = \vec{r}_p - \vec{r}_\gamma}$ for every point in $\Omega(t)$. This vector characterizes the evolution of the "shape" of Ω as the set moves along Γ. In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}_p$ vectors evolve.

h2. For every vector $\delta\vec{r}_p$, write an equation giving $\delta\vec{r}_p(t+\mathrm{d}t)$ in terms of $\delta\vec{r}_p(t)$ and the partial derivatives of $\vec{F}$ along Γ. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}_p(t)$ to $\delta\vec{r}_p(t+\mathrm{d}t)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(\mathrm{d}t)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t+\mathrm{d}t)$ and $\mathcal{A}(t)$. Note that:
(a) For linear transformations, hypervolumes are related by the absolute value of the determinant. †
(b) You need to calculate the determinant only up to $O(\mathrm{d}t)$ terms (neglect higher orders).
(c) For any square matrix $\boldsymbol{M}$, $\boldsymbol{\det(1+\epsilon\, M) = 1 + \epsilon\,\mathrm{tr}(M) + O(\epsilon^2)}$.
Optional: prove the formula in (c).

h4. Item **h3** will yield a formula of the form $\mathcal{A}(t+\mathrm{d}t) = \mathcal{A}(t) + (\text{something})\,\mathrm{d}t$. What differential equation is this?

† Multi-variable calculus: For a transformation $\boldsymbol{\vec{x} \to \vec{y}}$; $\boldsymbol{\int f(\vec{y})\,\mathrm{d}\vec{y} = \int f(\vec{y}(\vec{x}))|\det(J)|\,\mathrm{d}\vec{x}}$, where $\boldsymbol{J}$ = matrix of partial derivatives $\boldsymbol{\frac{\partial y_m}{\partial x_n}}$. Thus for an infinitesimal hypervolume $\boldsymbol{\delta V \to |\det J|\,\delta V}$. If $\boldsymbol{\vec{y} = M\,\vec{x}}$ (linear transformation), $\boldsymbol{J = M}$.

2 Neglect terms in equations (inertia in a forced-damped oscillator)

Statement: Neglect terms in equations

When modeling physical systems it is useful to be able to estimate how important the various physical effects that bear on the problem are, so that only the physics that matters is included in the model. The "kitchen sink" approach leads to models with too many unknown "free" parameters, and complicated system of equations that are very hard to solve (even with a computer). Not to mention the fact that "numerically" solving a system of equations whose behavior you do not understand can easily lead to trouble.

There is no such thing as fool-proof software that can reliably solve problems the user does not understand. This is particularly true for problems that involve solving pde.

Dimensional analysis is a tool that can help to identify effects that (maybe) can be neglected. I say "maybe" because the fact that some effect appears to be small does not mean that it can be neglected — for example: boundary layers and shocks are related to physical terms that a naive analysis classifies as "small". But a necessary condition to be able to neglect an effect is that it be small. Beyond this, hard thinking is needed. There is no magic bullet.

Here we will consider a *very simple example, basically: a toy model for a shock absorber.* Thus, imagine a mass $\boldsymbol{m}$, attached both to a spring (with spring constant $\boldsymbol{k > 0}$) and a damper (with damping constant $\boldsymbol{\nu > 0}$). We assume that the damper produces a force opposing the motion and proportional to the velocity. We also assume motion in 1-D, and a Hook law spring. In addition, we will assume that an external, time varying force, is acting on the mass.

Let $\boldsymbol{x}$ be the deviation of the mass from its equilibrium position (where the spring force vanishes). Then Newton's laws of motion involve four terms: (1) *inertia*, $\boldsymbol{m\,\ddot{x}}$, (2) *spring force*, $\boldsymbol{-k\,x}$, and (3) *damping force*, $\boldsymbol{-\nu\,\dot{x}}$. (4) *applied force*, $\boldsymbol{f = f(t)}$. Now **answer/perform the following questions/tasks:**

q1. What dimensions do k and ν have?

q2. The balance between damping and the spring force produces a characteristic time scale, $\boldsymbol{\tau_d}$. Write a formula for τ_d — **without solving any equations**.

Assume that the external force has the form $\boldsymbol{f = f_0\,F(t/\tau_d)}$, where f_0 is a typical force, and F is an a-dimensional function, with both the function and derivatives of size $O(1)$.

q3. The balance between inertia and the spring also produces a characteristic time scale, $\boldsymbol{\tau_s}$. Write a formula for τ_s — **without solving any equations**.

q4. The balance between the spring force and the applied force produces a characteristic length scale, $\boldsymbol{L}$. Write a formula for L — **without solving any equations**.

q5. Write the equation for the mass spring system using a-dimensional variables. Specifically: write the equation in terms of the variables $\tilde{\boldsymbol{x}} = \boldsymbol{x}/\boldsymbol{L}$ and $\tilde{\boldsymbol{t}} = \boldsymbol{t}/\boldsymbol{\tau_d}$. The equation can be written so that it involves a single a-dimensional number, $\boldsymbol{\epsilon}$, multiplying the second derivative. Write ϵ in terms of τ_d and τ_s.

Now **answer the questions:** What condition on ϵ is needed to be able to neglect the effects of inertia (i.e.: the term $m\,\ddot{x}$ in the equations) on the behavior? What does the condition mean in terms of the times τ_d and τ_s? Can you interpret the condition in terms of something being at, or almost, at equilibrium? Why is this a regime at which you would like a car shock-absorber to operate at?

q6. Note that τ_s is related to the period of oscillation, $T = \alpha\,\tau_s$, via some numerical constant α. **Find $\boldsymbol{\alpha}$** — you need to solve an equation to find α.

3 Dipole system

Statement: Dipole system

Task #1. Plot a computer generated phase plane portrait for the "Dipole system"

$$\dot{\boldsymbol{x}} = \boldsymbol{2\,x\,y} \qquad \text{and} \qquad \dot{\boldsymbol{y}} = \boldsymbol{y^2 - x^2}. \tag{3.1}$$

I strongly suggest that you use the PHPLdemoB MatLab script provided to you in the class website [MatLab toolkit].

Task #2. Find the critical points for this system, and linearize near them. What do the linearized equations tell you about the behavior near the critical points?

Task #3. Use the generated phase plane portrait to compute the index of the critical points.

Task #4. The generated phase plane portrait should suggest that the orbits for this system are circles. † In fact *any circle tangent to the y-axis at the origin would seem to be an orbit.* **Show that this is correct.**

† In MatLab, use "axis square" when plotting, so there are no distortions.

Hint for #3. Write a function $\boldsymbol{E}$ whose level curves are **all** the circles tangent to the y-axis at the origin, and show that $\boldsymbol{E}$ is conserved. You will not be able to obtain a function $\boldsymbol{E}$ without some singularity on the y-axis. ‡ The best you can do is have $\boldsymbol{E}$ singular at the origin only. This is related to the fact that the y-axis is the circle tangent to the y-axis at the origin, whose radius is infinity; while the origin itself corresponds to a zero radius.

‡ This is not a problem, since the y-axis can be easily analyzed separately.

4 Phase Plane Center Question #01

Statement: Phase Plane Center Question #01

Consider the equation

$$\ddot{\boldsymbol{x}} + (\boldsymbol{1} - \cos(\dot{\boldsymbol{x}})) + \boldsymbol{x} = \boldsymbol{0}. \tag{4.1}$$

This equation has a critical point at $\boldsymbol{x} = \dot{\boldsymbol{x}} = \boldsymbol{0}$, which is a center for linearized analysis **(show this)**. Is it a center for the full nonlinear equation as well? **Justify your answer** with an analytical argument. Finally: **plot a computer generated phase plane portrait for the system** (use the PHPLdemoB MatLab script provided with the MatLab toolkit in the course web page). Examine the phase plane in a reasonably large region enclosing the critical point; specifically: $-4 < x < 1$ and $-2.5 < \dot{x} < 2.5$. *What do you see? Can you guess what the complete phase plane looks like from this picture? How can you check if your guess is correct?*

5 Reversible system #01 (show reversible and sketch phase portrait)

Statement: Reversible system #01

Show that the system $\ddot{x} + x\,\dot{x} + x = 0$ is reversible and sketch the phase portrait. You can use a computer to do the picture, but you must justify the plot; in particular: include an analysis of any fixed point that occurs.

6 A system both gradient and Hamiltonian

Statement: A system both gradient and Hamiltonian

Consider the system $\dot{x} = \cos x \cosh y = f$ and $\dot{y} = \sin x \sinh y = g$.

a. Then $f_y = g_x$, hence this is a gradient system.[1] **Find the potential $V = V(x,\, y)$.**

b. Show that the system is also Hamiltonian, and find the Hamiltonian $H = H(x,\, y)$.

7 Two closed orbits enclosed by a third

Statement: Two closed orbits enclosed by a third

Consider a phase plane system, $\dot{x} = f(x,\, y)$ and $\dot{y} = g(x,\, y)$, where f and g are nice and smooth. Imagine that the system has two disjoint closed orbits (say: Γ_1 and Γ_2), and a third one (say: Γ_3) that encloses both.† **What is the minimal number of critical points that the system can have, and what are their indexes?**

† Example: Γ_1 = radius 1 circle centered at $(2,\, 0)$, Γ_2 = radius 1 circle centered at $(-2,\, 0)$, Γ_3 = radius 4 circle centered at $(0,\, 0)$.

8 Find a conserved quantity #01 (and sketch phase portrait)

Statement: Find a conserved quantity #01 (and sketch phase portrait)

Find a conserved quantity for the system $\ddot{x} = a - e^{x}$, and sketch phase portraits characteristic of the cases $a < 0,\ a = 0,\ a > 0$. Include an analysis of any fixed point that occurs.

THE END.

[1] Why? Can you show this?