Problem Set Number 03, (18.353/12.006/2.050)j MIT (Fall 2023)

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1 DiAn02. Speed of a non-linear diffusion front ($\nu \propto \mathcal{C}^2$)

Statement: Speed of a non-linear diffusion front

Consider some substance diffusing in an **isotropic**[1] **medium at rest.**
Then the following equation applies

$$\mathcal{C}_t = \operatorname{div}(\nu \, \nabla \mathcal{C}), \tag{1.1}$$

where

1. Fick's law of diffusion is assumed: the substance flux due to diffusion is along the gradient of the concentration, from higher to lower concentrations. For this it is important that the medium be isotropic — else the flux diffusion may occur along directions that depend both on the gradient of the concentration, as well as special directions in the media.
2. $\mathcal{C} = \mathcal{C}(\vec{x},\, t)$ is the substance concentration (mass per volume) — e.g.: grams per liter.
3. $\nu > 0$ is the *diffusion coefficient.*
4. $\nabla \mathcal{C}$ is the gradient of the concentration, and div denotes the divergence of a vector field.

When ν is a constant (1.1) reduces to the *linear diffusion equation*

$$\mathcal{C}_t = \nu \, \Delta \, \mathcal{C}, \tag{1.2}$$

where Δ is the *Laplace operator*, $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$.

However, there are situations where the diffusion coefficient is **not** constant, and depends (for example) on the concentration itself: $\nu = \nu(\mathcal{C})$. Here we will consider the particular case where $\nu \propto \mathcal{C}^2$ and the medium is homogeneous.[2] Then (1.1) reduces to

$$\mathcal{C}_t = \mu \operatorname{div}\left(\mathcal{C}^2 \, \nabla \mathcal{C}\right), \tag{1.3}$$

where **$\mu > 0$ is a constant.**

Under conditions where (1.3) applies, imagine that at $t = 0$ there is a very tiny blob of substance somewhere in the media. Then, due to the diffusion, the size of the blob will increase with time — with a sharp edge between the region where $\mathcal{C} > 0$, and the region where there is no substance. *Note: In this the behavior of (1.3) differs from that of (1.2). In the case of the linear diffusion equation, the blob's edge ceases to be sharp for $t > 0$, even if the initial blob has a sharp edge.*

Problem Tasks. What you are expected to do.

Let **M be the total mass in the blob,** and make the approximation that, **at time $t = 0$ the blob is just a point** — i.e.: all the substance's mass is concentrated in a blob so tiny that you can think of it as a point. Then perform the tasks below, ***using qualitative (but precise) physical arguments only*** — *do not solve the equation.*

[1] **Isotropic** means that the properties of the medium are invariant under rotation.
[2] **Homogeneous** means that the properties of the medium are the same everywhere.

q1. What dimensions does μ have?

q2. Argue that the shape of the blob is a sphere for $t > 0$.

q3. Find a formula for the radius of the blob $R = R(t)$ as a function of time — namely: a formula of the form $R = \alpha\, f(t)$, where α is a dimension-less constant (a number), and $f(t)$ is some function of time. You will not able to calculate α without solving the p.d.e. (1.3), which you are not expected to do. But you should be able to fully determine the function $f(t)$.

Assume now that $\boldsymbol{R(t) = 5}$ **mm** when $\boldsymbol{t = 3.75}$ **sec.**

q4. What value does R take for $t = 960\,\text{sec} = 16\,\text{min}$?

q5. For what value of t is $R = 5\,\text{cm}$?

q6. Would your answers change if the nonlinear diffusion occurred in the plane, instead of in 3-D? In particular, what are the answers to **q1** and **q3** in 2-D?

Hint. The formula giving the radius as a function of time must involve physical constants that allow it to transform time into length. These physical constants must result from the physical constants in the problem, and only them — e.g.: if you write a formula that involves the speed of light in it, something went wrong with your reasoning!

2 Overdamped pendulum with a torsional spring

Statement: Overdamped pendulum with a torsional spring

Suppose that the overdamped pendulum is connected to a torsional spring. As the pendulum rotates, the spring winds up and generates an opposing torque $-k\,\theta$, $k > 0$. The equation of motion is then $\boldsymbol{b\dot{\theta} + mgL\sin\theta = \Gamma - k\theta}$, where b is the damping coefficient.

a. Does this equation give a well-defined vector field on the circle?

b. Nondimensionalize the equation.

c. What does the pendulum do in the long run?

d. Show that many bifurcations occur as k is varied from 0 to ∞. What kind of bifurcations are they?

3 Exponential to algebraic decay transition

Statement: Exponential to algebraic decay transition

Consider the equation[3] $\boldsymbol{\dot{x} = -r\,x - x^3}$, with initial data $\boldsymbol{x(0) = x_0}$. *In the limit* $\boldsymbol{r \downarrow 0}$*, the solution to this equation transitions from exponential decay as* $\boldsymbol{t \to \infty}$ *(when* $\boldsymbol{r > 0}$*) to algebraic decay when* $\boldsymbol{r = 0}$*.*

Why? Because when x becomes very small, the behavior of the solution is controlled by the term $-r\,x$ if $r > 0$. On the other hand, if $r = 0$, the exact solution $x = x_0/\sqrt{1 + 2\,x_0^2\,t}$ decays like $1/\sqrt{2\,t}$. However, how do we know that the solutions decay when $r > 0$? Because then $\dot{x}/x = -(r + x^2) < r$, so that the solution size is less than $|x_0|\,e^{-r\,t}$.

Assume that $\boldsymbol{0 < r \ll 1}$. Then the solution should decay exponentially fast as $t \to \infty$, yet (because r is small) the solution should also be "close" to the solution to the problem when the term $-r\,x$ is neglected. That is: (3.1).

$$x_* = \frac{x_0}{\sqrt{1 + 2\,x_0^2\,t}} \tag{3.1}$$

How can these two things happen simultaneously?

[3] Note that this equation is one of the two possible normal forms for pitchfork bifurcations.

(a) Use separation of variables (or any other method) to solve $\dot{x} = -r\,x - x^3$, $x(0) = x_0$, $r > 0$, analytically. *Hint: what equation does $y = 1/x^2$ satisfy? Once you know y, finding the sign of x is easy.*

(b) For the solution in item **a**, show that $x \sim C\,e^{-r\,t}$ for $t \to \infty$, where C is a constant that you should compute.

(c) For the solution in item **a**, show that $x \sim x_*$ for $0 \leq r\,t \ll 1$, where x_* is as in (3.1). Notice that, as $r \downarrow 0$, the time interval over which this is valid gets larger and larger.

(d) To get some intuition on what is going on, plot the exact solution you obtained in item **a** versus x_* in (3.1) [plot #1]. In addition, plot the exact solution you obtained in item **a** versus the approximation in item **b** [plot #2]. Use the same parameter values for both plots, but different time ranges. *Suggestion: Use $x_0 = 1$ and $r = 0.01$. Then, for plot #1 use the range $0 < t < 50$, and for plot #2 the range $0 < t < 100$.*

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4 Attracting and Liapunov stable

Statement: Attracting and Liapunov stable

Recall the *definitions for the various types of stability* that concern critical points:
Let $\mathbf{x}^*$ be a fixed point of the system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then:

1. $\mathbf{x}^*$ is **attracting** if there is a $\delta > 0$ such that $\lim_{t\to\infty} = \mathbf{x}^*$ whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. That is: any trajectory that starts within δ of $\mathbf{x}^*$ *eventually* converges to $\mathbf{x}^*$. Note that trajectories that start nearby $\mathbf{x}^*$ *need not stay close in the short run*, but *must approach $\mathbf{x}^*$ in the long run.*
2. $\mathbf{x}^*$ is **Liapunov stable** if for each $\epsilon > 0$, there is a $\delta > 0$ such that $\|\mathbf{x}(t) - \mathbf{x}^*\| < \epsilon$ for $t > 0$, whenever $\|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$. Thus, trajectories that start within δ of $\mathbf{x}^*$ stay within ϵ of $\mathbf{x}^*$ for all $t > 0$.
   In contrast with attracting, Liapunov stability requires nearby trajectories to remain close *for all $t > 0$*.
3. $\mathbf{x}^*$ **is asymptotically stable** if it is *both* attracting and Liapunov stable.
4. $\mathbf{x}^*$ is **repeller** if there exist $\epsilon > 0$ and $\delta > 0$ such that: if $0 < \|\mathbf{x}(0) - \mathbf{x}^*\| < \delta$, then (after some critical time) it will be $\|\mathbf{x}(t) - \mathbf{x}^*\| > \epsilon$ (i.e., for $t > t_c$). Repellers are a special kind of *unstable* critical points.

For each of the following systems, decide whether the origin is attracting but not Liapunov stable, Liapunov stable, asymptotically stable, repeller, or unstable but not a repeller.

a) $\dot{x} = 2\,y$ and $\dot{y} = -3\,x$.
b) $\dot{x} = y\,\cos(x^2 + y^2)$ and $\dot{y} = -x\,\cos(x^2 + y^2)$.
c) $\dot{x} = -x$ and $\dot{y} = -|y|\,y$.
d) $\dot{x} = 2\,x\,y$ and $\dot{y} = y^2 - x^2$. *Hint: what happens along $x = 0$?*
e) $\dot{x} = x - 2\,y\,x^2 - 4\,y^3$ and $\dot{y} = y + x^3 + 2\,x\,y^2$.
f) $\dot{x} = y$ and $\dot{y} = x$.
g) Finally, consider the critical point $(x,\,y) = (1,\,0)$, for the system

$$\dot{x} = (1 - r^2)\,x - (1 - \tfrac{x}{r})\,y \quad \text{and} \quad \dot{y} = (1 - r^2)\,y + (1 - \tfrac{x}{r})\,x, \tag{4.1}$$

defined in the "punctured" plane $r = \sqrt{x^2 + y^2} > 0$. *Hint: write the equations in polar coordinates.*

Additional hints. In some cases you can get the answer by finding a function $\mathcal{J} = \mathcal{J}(x,\,y)$ with a local minimum at the origin such that $\frac{d\mathcal{J}}{dt} > 0$ along trajectories — or maybe one such $\frac{d\mathcal{J}}{dt} < 0$, or maybe one such $\frac{d\mathcal{J}}{dt} = 0$. In other cases look for special trajectories that either leave, or approach, the origin.

THE END.