Problem Set Number 02, (18.353/12.006/2.050)j
MIT (Fall 2023)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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Contents

1 Blow up in finite time #01

Statement: Blow up in finite time #01

Consider the evolution of $x(t)$ according to $\dot{x} = rx + x^3$, where $r > 0$ is fixed.

a. Show that the origin is an unstable fixed point.

b. For $x(0) = x_0 \neq 0$, show that $|x(t)| \to \infty$ as $t \to t_0$, where

$$t_0 = \frac{1}{2r} \log\left(1 + \frac{r}{x_0^2}\right).$$

2 Leaky Bucket by dimensional analysis

Statement: Leaky Bucket by dimensional analysis

Consider a cylindrical bucket of cross-sectional area A, with a small hole at the bottom (of cross-sectional area $a \ll A$), filled with water up to a depth $h = h(t)$ (h is a function of time because the bucket is leaking water). The **objective** is to write an equation for h, $\dot{\boldsymbol{h}} = \boldsymbol{f(h)}$, under the following assumptions:

1. The leaking is very slow, so that we can neglect any fluid motion within the bucket.
2. The small hole at the bottom is large enough that it is safe to neglect surface tension — thus the leaking occurs though a small continuous stream (jet) out of the hole, rather than a "drip-drip" involving drops.

3. We now argue that *the speed at which the bucket empties* (i.e.: $\dot{h}$) *depends on A and a through their quotient a/A only, and it is proportional to it.* That is

$$\dot{\boldsymbol{h}} = (\boldsymbol{a}/\boldsymbol{A})\,\boldsymbol{F}(\boldsymbol{h}), \tag{2.1}$$

where F depends on neither a nor A.

Argument: for a given water flux through the small hole, it is clear that $\dot{h}$ is proportional to $1/A$. On the other hand, the flux through the small hole is proportional to the pressure difference across the hole, times a — where the pressure difference depends on h, g, etc., but **not** A. A somewhat simpler argument is: n equal buckets will empty at the same rate; but this is the same as multiplying both A and a by n (this works for integers only, though).

Given the assumptions above, $\boldsymbol{F}$ in (2.1) should **depend only** on the following dimensional quantities: $\boldsymbol{g}$ (the acceleration of gravity), $\boldsymbol{\rho}$ (water density), and (of course) $\boldsymbol{h}$.
Task: use dimensional analysis to *determine F up to a multiplicative constant.*[1] A bit of extra thinking will also tell you *what is the sign of F.*

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3 Get equation from phase line portrait problem #07

Statement: Get equation from phase line portrait problem #07

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{3.1}$$

where f is some function which has a continuous derivative. Assume that (3.1) has at least two critical points, and let $x_1 < x_2$ be two consecutive critical points (there is no other critical point between them). Assume now that both x_1 and x_2 are stable. **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

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4 Get equation from phase line portrait problem #08

Statement: Get equation from phase line portrait problem #08

Consider the ode on the line

$$\frac{dx}{dt} = f(x), \tag{4.1}$$

where f is some function which has a continuous derivative. Assume that (4.1) has infinitely many critical points $-\infty < \dots x_n < x_{n+1} < \dots \infty$. Assume that all the critical points are semi-stable.[2] **Is this possible? Does a function $f = f(x)$ yielding this exist?**
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

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[1] Dimensional analysis alone cannot distinguish F from $2F$, or from πF, or from $\dots$
[2] A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.

5 Implicit function problem #02

Statement: Implicit function problem #02

Consider the following equation
$$\boldsymbol{f(x,\,\lambda) = x + \lambda\,\cos(x) + \lambda^3 = 0}, \tag{5.1}$$
with the particular solution $\boldsymbol{(x,\,\lambda) = (0,\,0)}$. Since $f_x(0,\,0) = 1$, the implicit function theorem guarantees that: *there is a neighborhood of $\lambda = 0$ where (5.1) has a unique solution, $x = X(\lambda)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function

of λ. This means that X has a Taylor series
$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\,\lambda^n}, \tag{5.2}$$
which converges for $|\lambda|$ small enough. **Find x_1, x_3, x_5, and x_n for all even n.**

Hint #1: Since X is small, write the cosine in (5.1) as a power series. Next substitute (5.2) into (5.1) and reorganize the result in terms of powers of λ. Finally make the coefficient of each power of λ vanish.[3] *This will give you equations that will allow you to solve for x_0 first, then x_1, and so on.*

Hint #2: There is a special property of the equation that tells you what x_n is for n even. ***Figure this out first,*** *it will save you a lot of pointless algebra. This problem involves much less calculation than you might think.*

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6 Inverse function problem #01

Statement: Inverse function problem #01

Consider the following equation
$$\boldsymbol{y = x + \sin(x) = f(x)}, \tag{6.1}$$
where $\boldsymbol{f(0) = 0}$ and $\boldsymbol{f'(0) = 2 \neq 0}$. The inverse function theorem guarantees that *there is a neighborhood of $x = 0$ where f has a unique inverse, $x = X(y)$, such that $X(0) = 0$.* Furthermore, since f is an analytic function, X is an analytic function. This

means that X has a Taylor series
$$\boldsymbol{X = \sum_{n=0}^{\infty} x_n\,y^n}, \tag{6.2}$$
which converges for $|y|$ small enough. **Find x_1, x_3, x_5, and x_n for all even n.**

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7 Phase line portrait problem #02

Statement: Phase line portrait problem #02

Consider the following ode on the line
$$\boldsymbol{\frac{dx}{dt} = f(x) = f(x) = x^2 - x^3}. \tag{7.1}$$
Draw its phase line portrait, indicating the critical points, and their stability properties.

In addition: describe *quantitatively* the behavior of the solutions near the critical points (i.e.: at what rate do they approach or leave them), as well as the behavior of the solutions when $|x|$ is large. **In particular:** *Are there solutions that cease to exist for some finite value of t, or are the solutions valid for all times?* If some solutions are defined for all times, and others are not: *state a condition that guarantees that a solution is defined for all times* — e.g.: if $x(t)$ is in some range/interval.

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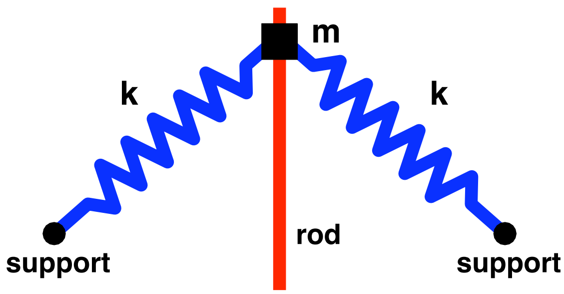
[3] You only need to carry this calculation up to powers λ^n with $n \leq 5$.

8 Toy model for shell buckling

Statement: Toy model for shell buckling

Hold a ping-pong ball between your thumb and index fingers and squeeze it. If you do not apply enough force, the ball will deform slightly with a purely elastic response. But, if you push hard enough, the ball will buckle and you will make a (permanent) dent on it — and the ball will be ruined. This is the phenomena of (thin) shell buckling.

Shell buckling is a very rich phenomena,[4] way beyond the scope of this course. Here we will study an extremely simplified (1-D) version of this phenomena (the emphasis here being on "toy" model) where all the geometrical richness of the original setting is gone, and only the buckling bifurcation remains.



A bead of mass m (black square) can slide along a rigid vertical rod (in red). The bead is connected by two equal springs (in blue), with spring constant k, to two supports placed symmetrically on each side of the rod. See the text for further details.

Figure 8.1: Toy model for shell buckling.

A sketch depicting the model is shown in figure 8.1. Further assumptions and notation are:

1. Idealize the bead as a point mass.

2. Let x be the vertical distance, along the rod, of the bead from the horizontal line joining the spring supports. Let $x > 0$ if the bead is above the supports and $x < 0$ if below.

3. Let $\boldsymbol{h > 0}$ be the distance of the spring supports from the rod, and let $\boldsymbol{L > 0}$ be the springs equilibrium length. **Assume $\boldsymbol{L > h}$**, so that the springs are under compression for $x = 0$.

4. Hook's law applies to the springs. Thus they exert a force of magnitude $F = k\,(\ell - L)$, where ℓ is the spring length, along the spring axis, pushing if $\ell < L$, and pulling if $\ell > L$.

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\,\dot{x}$, where $b > 0$ is a constant.

6. Because the rod is rigid, we need to consider only the vertical components of the various forces that act on the bead. These forces are: (i) Gravity, of magnitude $m\,g$, pointing down. (ii) The forces by the springs. (iii) Friction along the rod. *Note that here we assume that the force gravity is significant, so that there is no up-down symmetry in this problem.*

PROBLEM TASKS:

A. *Derive an ode for the bead position, and write it in appropriate a-dimensional variables.*[5]

B. *Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be "large"?*

C. *Analyze the bifurcations that occur for the equation resulting from item B, as the bead mass changes — in this toy model, increasing the bead mass plays the role of squeezing harder on the ping-pong ball. What type of bifurcation(s) occur?*

Hint: It is a bad idea to try to do this by attempting to solve for the critical points and bifurcation thresholds analytically. A qualitative, graphical, analysis is the best way to go.

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[4] Lots of interesting and important questions arise. For example: What is the shape of the dent that forms? The dent's edges have sharp corners: why these corners form, and how do they propagate as further pressure is applied?

[5] Suggestion: to a-dimensionalize use h for length and $b/(2\,k)$ for time.

D. *The picture in figure* 8.1 *corresponds, in this toy model, to the ping-pong ball in a more-or-less spherical shape. What is the "buckled" state?*

E. *What a-dimensional parameter controls when bifurcations happen?* This under the assumption:

$$\textbf{The ratio } \gamma = L/h > 1 \textbf{ is kept fixed.} \tag{8.1}$$

Thus γ is **not** the bifurcation parameter to use; something else is.

THE END.