Problem Set Number 01, (18.353/12.006/2.050)j
MIT (Fall 2023)

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Due September 19, 2023 (turn it in via the canvas course website). Note that this is an updated version of the September 8 original, fixing various typos and with changes that make the problem easier (I hope). Please read again the description at the top of the MatLab script, because I have also added improvements there.

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1 Computer exercises with a 1-D map

Statement: Computer exercises with a 1-D map.

The objective of this problem is to give an elementary first introduction to concepts such as: fixed point, stability, bifurcations, and chaos, via "experimental" (i.e., numerical) computation. We do this by using a very simple mathematical model. This model is highly abstract, but we will argue later in the course that the model is also (to a significant degree) representative of the behavior of many real systems. In addition, this assignment also introduces the kind of computing problems we shall often, but not always, assign.

In this model we will assume that the system is described by a single time-dependent variable, $x(t)$. This variable could represent, say, the globally averaged temperature on the Earth's surface, the size of a particular population of animals on some secluded island (or in a Petri dish), some particular stock market average, etc. Furthermore, we will suppose that we are interested only in the values $x_n = x(t_n)$ at discrete times $t_n = n\,\Delta t$, where Δt is some suitable interval of time (say, a day). The evolution of x in time may then be written as

$$x_{n+1} = F(x_n). \qquad (1.1)$$

where F is some function that describes the dynamics. For any of the examples mentioned above, it is obvious that the "true" F would involve complicated equations. Rather than going into that kind of detail, we consider below two rather simple choices for F. That is:

$$F(x) = \mu\, x, \qquad (1.2)$$

and

$$F(x) = c\,\pi\,\sin(x), \qquad (1.3)$$

where μ and c are (real) constants. **These are your tasks:**

1. For (1.1–1.2), without using the computer, what can you predict about the behavior of x_n, given an initial condition $x_0 \neq 0$? (Note: of particular interest is the behavior for $n \to \infty$).

1a. Qualitatively, what is different about the cases $|\mu| < 1$, $|\mu| > 1$, and $|\mu| = 1$?

1b. What is the qualitative difference between the evolution with $\mu > 0$ and $\mu < 0$?

1c. Check your conclusions numerically (do not hand in any plots for this, you can even use a calculator).

Note: In this case $x = 0$ is a *fixed point* of the evolution (i.e.: $x_{n+1} = x_n$ if $x_n = 0$), and your conclusions, in particular, speak to the *stability* of this fixed point (what happens if a small perturbation is applied).

2. We now turn to (1.1–1.3). For this example a MatLab script, IterateSineMap, is supplied. Please **read the script description at the top of the IterateSineMap.m file**, which describes in detail what the script does and how to use it. **The aim of this part of the problem is to explore the behavior of (1.1–1.3) as the parameter c changes**. Notice that, **if $0 \leq c \leq 1$, then for $0 \leq x_n \leq \pi$, we have $0 \leq x_{n+1} \leq \pi$.** Thus we will restrict the exploration to these ranges and ask **How does x_n behave for n large?** (IterateSineMap was specifically designed for this task).

The system (1.1–1.3) exhibits many behaviors. For example, x_n may approach a constant (a fixed point) as $n \to \infty$, or it may approach a *periodic cycle*, where $x_{n+p} = x_n$ (p is the period), or the $n \to \infty$ behavior may be chaotic (see remark **1.1**), or it may exhibit *intermittent chaos*, where the sequence alternates (seemingly randomly) between being close to a periodic cycle, and chaotic bursts.[1] These various behaviors correspond to different values of c, with the transition from one to another occurring at *critical values* of c (of which there are many; infinitely many, in fact).

A value $c = c_c$ is called a critical value if the long term evolution for x_n changes *qualitatively* as c crosses c_c. These qualitative changes are called **bifurcations**. For example, on one side of c_c the system may be attracted to some fixed point x_*, and to a different fixed point on the other side. Or maybe the behavior switches from fixed point to periodic of some period $p > 1$, or the period changes across c_c, etc.

These are the questions you are asked to investigate:

2a. What happens for $0 \le c < c_1 = 1/\pi \approx 0.3183$, and for $c_1 < c < c_2$, where[2] $c_2 \approx 0.72$. Can you explain why c_1 happens at $1/\pi$? *Hint: Linearize the map for x small, and use part* **1**.

Note that c_1 and c_2 are *critical values, or bifurcation points.*

2b. What happens for c slightly above c_2? *Hint: Check $c = 0.73$ first, and then explore more carefully.*

2c. In fact, there is an infinite sequence of critical points $c_1 < c_2 < c_3 < \dots$, with $\lim_{n\to\infty} c_n = c_\infty$ (where $c_\infty \approx 0.8655791$). In each "window" $c_n < c < c_{n+1}$, the limit behavior of x_n is very simple — in part **2b** you should already have discovered what happens for $c_2 < c < c_3$.
Compute the first few c_n, say, c_3 and c_4, **and describe the behavior in the corresponding windows,** say $c_3 < c < c_4$ and $c_4 < c < c_5$ — note that you do not need to know c_5 to ascertain the behavior in $c_4 < c < c_5$. **Provide values with, at least, 4 accurate decimal digits: $c_n \approx 0.abcd$.**

Approximate values are as follows: $c_3 \approx 0.805$, $c_4 \approx 0.850$, $c_5 \approx 0.863$, and $c_6 \approx 0.865$. Note that beyond c_8 4 digits are not enough to tell the values apart, so the calculations get increasingly harder.

2d. From your results in **2c**, **can you guess what the pattern is for $c_n < c < c_{n+1}$.**

2e. Now look at what happens slightly beyond c_∞. Specifically, **what do you see for $c = 0.866$?** Do your best here; see remark **1.1**.

2f. **What do you see for $c = 0.880876$?**

2f. **What do you see for $c = 0.8814$?** Note: in fact, the behavior you see here happens for a whole window, $0.880877 \dots < c < 0.881464 \dots$.
Optional: keep going a little beyond $c = 0.881464 \dots$, what happens?

2g. **What do you see for $c = 0.9395$?** Note: in fact, the behavior you see here happens for a whole window, $0.937819 \dots < c < 0.940943 \dots$.
Optional: keep going a little beyond $c = 0.940943 \dots$, what happens?

Remark 1.1 *For this problem we will not define chaos (will be done later). Here simply check that the behavior is not periodic of any "reasonable" period,*† *e.g.: $1 \le p \le 16$ (many can be excluded by simple eye-sight).* ♣

† Because you are doing this in a computer, there is only a finite number of values x_n can take, so the computed sequence x_n will always be periodic ... but the period can be huge, for any practical purposes "infinite".

THE END.

[1] The MatLab script provided allows you to check for these behaviors.

[2] A more precise value is $c_2 = 0.719961682979535\dots$ We will see how this can be computed later in the semester.