1 Ellipsoidal trapping region for the Lorenz equations

Statement: Ellipsoidal trapping region for the Lorenz equations

Show that the ellipsoidal region \( E \), defined by
\[
E(x, y, z) = r x^2 + \sigma y^2 + \sigma (z - 2r)^2 \leq C,
\]
for some constant \( C \), is trapping and attracting:
all the trajectories of the Lorenz equations eventually enter \( E \), and stay there forever. Recall that the equations are
\[
\dot{x} = \sigma (y - x), \quad \dot{y} = rx - y - xz, \quad \text{and} \quad \dot{z} = xy - bz, \quad \text{where} \quad \sigma, r, b > 0.
\]
Hint: Calculate \( \dot{E} \), then find an ellipsoidal region \( \Omega \) such that \( \dot{E} < -\delta \) outside \( \Omega \), where \( \delta > 0 \) is some constant. Then show that defining \( C \) as the maximum value of \( E \) over \( \Omega \) works.

2 Justify the bead on a wire reduction

Statement: Justify the bead on a wire reduction

In Strogatz’ book, “bead on a wire” problems are often modeled by equations of the form
\[
\dot{x} = f(x).
\]
(2.1)
A more complete description of these systems typically has the form
\[
m \frac{d^2X}{dT^2} + b \frac{dX}{dT} = F f \left( \frac{1}{L} X \right),
\]
(2.2)
where \((m, b, F, L)\) are physical constants,\(^1\) and \(f\) has no dimensions (\(T\) is time, with dimensions).

\(^1\) For example: mass, damping coefficient, typical applied force, and typical length scale. But they can also be related to the capacitance, resistance, inductance and applied potential in a circuit. Or some other physics.
1. Introduce appropriate variables with no dimensions, and re-write (2.2) in terms of them. Then use phase plane analysis to fully justify a condition under which (2.1) applies. Hint: the process here is very similar to the one used to analyze relaxation oscillations. That is, show that there is an attracting curve in the phase plane, which solutions approach very fast.

2. Estimate the size of the error in (2.1). That is, we can write $\dot{x} = f(x) + E$, where $E$ is the error. Find a leading order estimation for $E$.

3. Multiple scales and limit cycles #03

Statement: Multiple scales and limit cycles #03

Consider the nonlinear oscillator described by the equation, for $x = x(t)$,

$$\frac{d^2 x}{dt^2} - \epsilon (1 - x^4) \frac{dx}{dt} + x = 0,$$

where $0 < \epsilon \ll 1$. This system has a limit cycle: compute its approximate amplitude and period using the method of multiple scales (also known as two-timing). The idea is to adapt the method presented in lectures for determining the amplitude of the limit cycle for the van der Pol oscillator.

Start by replacing the “single-time” dependence in $x = x(t)$ by a “two-times” dependence $x = x(t, \tau)$, where $\tau = \epsilon t$ is the “slow-time” over which the parameters in the harmonic oscillator ($\epsilon = 0$ case) solutions approximating the solutions to (3.1) evolve. Then

1. Then (from the chain rule) $\frac{dx}{dt} \mapsto \frac{\partial x}{\partial t} + \epsilon \frac{\partial x}{\partial \tau}$ and $\frac{d^2 x}{dt^2} \mapsto \frac{\partial^2 x}{\partial t^2} + 2\epsilon \frac{\partial^2 x}{\partial t \partial \tau} + \epsilon^2 \frac{\partial^2 x}{\partial \tau^2}$ in (3.1).

2. Use the expansion $x = x_0(t, \tau) + \epsilon x_1(t, \tau) + O(\epsilon^2)$, to show that $(\partial_t x_0 + x_0) + \epsilon(\partial_t x_1 + x_1 + 2\partial_\tau x_0 + (x_0^4 - 1) \partial_t x_0) = O(\epsilon^2)$.

3. Solve the $O(1)$ system for $x_0$ real. Write your answer in terms of the complex amplitude $A(\tau)$.

4. Use your answer for $x_0$, and compute the quantities $2\partial_\tau x_0, x_0^4$ and $\partial_t x_0$ in terms of $A$. By considering the secular terms that arise in the $O(\epsilon)$ system, show that $A(\tau)$ must satisfy

$$A_\tau = \frac{1}{2} A - A |A|^2. \quad (3.2)$$

5. Write $A$ in polar form $A = r(\tau) e^{i \phi(\tau)}$, and find the differential equations satisfied by $r$ and $\phi$. Determine the stable fixed point of the radial equation.

6. Deduce that the limit cycle solution is (approximately) given by $x = 2^{3/4} \cos(t) + O(\epsilon)$.

4 Problem 07.03.x1 (True or false)?

Statement for problem 07.03.x1

True or false? There exists a phase plane system

$$\frac{dx}{dt} = f(x, y) \quad \text{and} \quad \frac{dy}{dt} = g(x, y), \quad (4.1)$$
where \( f \) and \( g \) are fairly smooth (say, second partial derivatives continuous), and a simply connected (not empty) closed and bounded region \( \mathcal{R} \), such that

A. \( \mathcal{R} \) is a trapping region for the system in (4.1).

B. The system in (4.1) has no critical points in \( \mathcal{R} \).

If true, give an example. If false, prove it so.

Recall that an open set is simply connected if it has no holes. The technical definition is: any loop (closed continuous curve) contained in the set can be continuously deformed into a point, while remaining inside the set.

Hint. Index theory and Poincaré Bendixon may help.

5 Volume evolution

Statement: Volume evolution

Consider some arbitrary orbit, \( \Gamma \), for the system

\[
\frac{dr}{dt} = \vec{F}(\vec{r}), \quad \text{where} \quad \vec{r} \quad \text{and} \quad \vec{F} \quad \text{are vectors in} \quad \mathbb{R}^n,
\]

and \( \vec{F} \) has continuous partial derivatives up to (at least) second order. That is: \( \Gamma \) is the curve in \( \mathbb{R}^n \) given by some solution \( \vec{r} = \vec{r}_\gamma(t) \) to (5.1). Then

A. Let \( \Omega = \Omega(t) \) be an “infinitesimal” region that is being advected, along \( \Gamma \), by the flow given by (5.1). For example:

A1. Let \( \Omega(0) \) be a ball of “infinitesimal” radius \( dr \), centered at \( \vec{r}_\gamma(0) \).

A2. For every point \( \vec{r}_p^0 \in \Omega(0) \), let \( \vec{r} = \vec{r}_p(t) \) be the solution to (5.1) defined by the initial data \( \vec{r}_p(0) = \vec{r}_p^0 \).

A3. At any time \( t_\ast \), the set \( \Omega(t_\ast) \) is given by all the points \( \vec{r}_p(t_\ast) \), where \( \vec{r}_p^0 \) runs over all the points in \( \Omega(0) \).

Note that \( \Omega(0) \) need not be a ball. Any infinitesimal region containing \( \vec{r}_\gamma(0) \) will do. All we need is that the notion of hypervolume applies to it — see item B.

B. Let \( A = A(t) \) be the hypervolume of \( \Omega(t) \). Note: (i) if \( n = 1 \) the hypervolume is the length; (ii) if \( n = 2 \) the hypervolume is the area; (iii) if \( n = 3 \) the hypervolume is the volume; etc.

Find a differential equation for the time evolution of \( A \).

Optional: use the differential equation to show that \( \det(e^{Bt}) = e^{t \text{tr}(B)} \) for any square matrix \( B \) — where \( \text{tr}(B) \) denotes the trace of \( B \). Note: what you are asked to do here is to do the proof using the differential equation, specifically — not by some other means, like (say) linear algebra.

Hints.

h1. Introduce the vector \( \delta \vec{r}_p = \delta \vec{r}_p(t) = \vec{r}_p - \vec{r}_\gamma \) for every point in \( \Omega(t) \). This vector characterizes the evolution of the “shape” of \( \Omega \) as the set moves along \( \Gamma \). In order to calculate how \( A(t) \) evolves, you only need to know how the \( \delta \vec{r}_p \) vectors evolve.

h2. For every vector \( \delta \vec{r}_p \), write an equation giving \( \delta \vec{r}_p(t + dt) \) in terms of \( \delta \vec{r}_p(t) \) and the partial derivatives of \( \vec{F} \) along \( \Gamma \). Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from \( \delta \vec{r}_p(t) \) to \( \delta \vec{r}_p(t + dt) \) given by a linear transformation. Make sure that this linear transformation correctly includes the \( O(dt) \) terms, which you will need to calculate time derivatives.
h3. From the transformation in item h2 derive a relationship between $A(t + dt)$ and $A(t)$. Use the fact that, for linear transformations, hypervolumes are related by the absolute value of the determinant. You need to calculate the determinant only up to $O(dt)$.

h4. Use the result in item h3 to calculate the time derivative of $A$, and obtain the differential equation.

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Problems motivated by Strogatz’s book

The problems below are either from Strogatz’s book, or are motivated by problems there. If different, do the version here, not the one in the book.

6 Problem 08.02.06 - Strogatz (Hopf bifurcation using a computer)

Statement for problem 08.02.06

For the following system

\[
\frac{dx}{dt} = \mu x + y - x^3 \quad \text{and} \quad \frac{dy}{dt} = -x + \mu y + 2y^3, \tag{6.1}
\]

a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of $\mu$, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius $R$ scales with $\mu$ as predicted by theory.

Hints: (a) The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence check the behavior of the orbits near the critical point for a very small value of $\mu$ to see if the nonlinear terms stabilize or de-stabilize. (b) In the subcritical case the limit cycle is unstable, hence hard to compute accurately forward in time. However, if you run the system backwards in time the solutions converge to the limit cycle!

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THE END.