Problem Set Number 05, (18.353/12.006/2.050)j
MIT (Fall 2021)

Due Fri October 29, 2021.
Turn it in via the canvas course website.

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Note: for problems from Strogatz’s book, if the version here differs from the one in the book ...................... do the version here.

1 Problem 05.01.02 - Strogatz (Asymptotic behavior as \( t \to \infty \))

Statement for problem 05.01.02

Consider the system \( \dot{x} = ax, \; \dot{y} = -y \), where \( a < -1 \). Show that all trajectories become parallel to the y-direction as \( t \to \infty \), and parallel to the x-direction as \( t \to -\infty \).

Hint: Examine the slope \( \frac{dy}{dx} = \dot{y}/\dot{x} \).

Strictly speaking, not all trajectories satisfy these two statements. What are the exceptions?

2 Problem 06.01.08 - Strogatz (Computer generated phase portrait)

Statement for problem 06.01.08

Plot a computer generated phase plane portrait for the van der Pol oscillator

\[
\frac{dx}{dt} = y, \quad \text{and} \quad \frac{dy}{dt} = -x + y(1 - x^2).
\]
3 Problem 21.10.15 - Dipole fixed point

Statement for problem 21.10.15

Plot a computer generated phase plane portrait for the “Dipole fixed point” system

\[
\frac{dx}{dt} = 2xy \quad \text{and} \quad \frac{dy}{dt} = y^2 - x^2.
\]  \hspace{1cm} (3.1)

**Extra Task #1.** Find the critical points for this system, and linearize near them. What do the linearized equations tell you about the behavior near the critical points?

**Extra Task #2.** Use the generated phase plane portrait to compute the index of the critical points.

**Extra Task #3.** The generated phase plane portrait should suggest that the orbits for this system are circles. In fact any circle tangent to the y-axis at the origin would seem to be an orbit. **Show that this is correct.**

**Hint for #3.** Write a function \( E \) whose level curves are the circles tangent to the y-axis at the origin, and show that \( E \) is conserved. You will not be able to avoid the fact that \( E \) will have singularities somewhere on the y-axis \(^2\) (the best you can do is having \( E \) singular at the origin). This is related to the fact that the y-axis is the circle tangent to the y-axis at the origin, whose radius is infinity — while the origin itself corresponds to a zero radius.

4 Problem 06.01.11 - Strogatz (Computer generated phase portrait)

Statement for problem 06.01.11

Plot a computer generated phase plane portrait for the “Parrot” system

\[
\frac{dx}{dt} = y + y^2 \quad \text{and} \quad \frac{dy}{dt} = -x + \frac{1}{5}y - xy + \frac{6}{5}y^2, \]

in some “large” square, say: \(-6 \leq x, y \leq 6\).

**A.** Note how all the orbits eventually turn back towards the origin, to form the “parrot’s eye.

**B.** Linearize the system near \((x, y) = (0, 0)\), and show that it is an unstable spiral.

**How are A and B consistent? What is going on near the eye?**

5 Problem 06.01.12 - Strogatz (Saddle connections)

Statement for problem 06.01.12

A certain system is known to have exactly two fixed points, both of which are saddles. Sketch phase portraits in which

**a.** There is a single trajectory that connects the saddles.

**b.** There is no trajectory that connects the saddles.

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1 In Matlab, use “axis square” when plotting so there are no distortions.

2 Not an issue, since the y-axis can be analyzed separately with ease.
6 Problem 07.02.06 - Strogatz
(Find the potential for a gradient system)

Statement for problem 07.02.06

Given that a system is a gradient system, here is how to find its potential function $V$. Suppose that $\dot{x} = f(x, y)$ and $\dot{y} = g(x, y)$. Then $\dot{x} = -\nabla V$ implies $f(x, y) = -\frac{\partial V}{\partial x}$ and $g(x, y) = -\frac{\partial V}{\partial y}$. These two equations may be "partially integrated" to find $V$. Use this procedure to find $V$ for the following gradient systems.

a) $\dot{x} = y^2 + y \cos(x)$, and $\dot{y} = 2x y + \sin(x)$.

b) $\dot{x} = 3x^2 - 1 - \exp(2y)$, and $\dot{y} = -2x \exp(2y)$.

7 Problem 16.10.06 Systems both potential and Hamiltonian

Statement for problem 16.10.06

Consider the system $\dot{x} = \cos x \cosh y = f$ and $\dot{y} = \sin x \sinh y = g$.

a. Then $f_y = g_x$, hence this is a gradient system. Find the potential $V = V(x, y)$.

b. Show that the system is also Hamiltonian, and find the Hamiltonian $H = H(x, y)$.

c. Show that the system has the complex form $\dot{z} = \cos \bar{z}$, where $z = x + iy$ and $\bar{z} = x - iy$.

THE END.

3 Why? See exercise 7.2.5 (part a).