Problem Set Number 04, (18.353/12.006/2.050)j
MIT (Fall 2021)
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October 13, 2021

Due Fri October 22, 2021.
Turn it in via the canvas course website.

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1 Classify fixed points #01 (linearize and find the fixed points type)

Statement: Classify fixed points #01
Consider the system
\[ \dot{x} = x(4 - y - x^2), \quad \dot{y} = y(x - 1). \]
Then
a. Find the fixed points.
b. Linearize the equation near each fixed point, and classify the fixed points (saddles, stable nodes, etc.).

2 Damped harmonic oscillator (non-dim.; write as system; classify phase portrait)

Statement: Damped harmonic oscillator
The motion of a damped harmonic oscillator is described by \( m \ddot{x} + b \dot{x} + k x = 0 \), where \( b > 0 \) is the damping constant.

a. Introduce a dimensionless time variable \( \hat{t} \) so that \( \hat{x}(\hat{t}) = x(t) \) evolves according to
\[
\frac{d^2 \hat{x}}{d\hat{t}^2} + 2 \mu \frac{d\hat{x}}{d\hat{t}} + \hat{x} = 0,
\]
where \( \mu > 0 \) is a dimensionless constant that you should define.
b. Rewrite (2.1) as a two-dimensional linear system.
c. Classify the fixed point at the origin and sketch the phase portrait. Be sure to show all the different cases that can occur, depending on the size of \( \mu > 0 \). Include the nullclines and eigendirections (when real) on your phase portrait.
d. How do your results relate to the standard notions of overdamped, critically damped, and underdamped vibrations?

3 Find a conserved quantity #02 (and sketch phase portrait)

Statement: Find a conserved quantity #02 (and sketch phase portrait)

Find a conserved quantity for the system $\ddot{x} + x \dot{x} + x = 0$, and sketch the phase portrait. Include an analysis of any fixed point that occurs.

Hint: first write the equation as a system for $x$ and $v = \dot{x}$. Then look at the equation for $dv/dx = \dot{v}/\dot{x}$, and note that it can be solved by separation of variables. The free constant in this solution is a conserved quantity!

Note: the conserved quantity is valid away from the line $v = -1$ only. Other than this fact, what else makes this line special?

4 Liapunov Function # 01

Statement: Liapunov Function # 01

Show that the system

$$\frac{dx}{dt} = -x + 2y^3 - 2y^4 \quad \text{and} \quad \frac{dy}{dt} = -x - y + xy,$$

has no periodic solutions.

Hint. Find a Liapunov function. Try the form $L = x^m + ay^n$.

5 Reversible system #02 (show reversible and sketch phase portrait)

Statement: Reversible system #02

Show that the system $\ddot{x} = a - e^x$ is reversible and sketch the phase portraits characteristic of the cases $a < 0$, $a = 0$, and $a > 0$. Include an analysis of any fixed point that occurs.

6 Reversible system #03 (a "strange" reversible system)

Statement: Reversible system #03

Consider the system where $E = E(x, y)$, $a = a(x, y)$, and

$$\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -a E_y - b E_x \\ a E_x - b E_y \end{pmatrix} = \begin{pmatrix} -b & -a \\ a & -b \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = A \nabla E,$$
Problems motivated by Strogatz’s book

The problems below are either from Strogatz’s book, or are motivated by problems there. If different, do the version here, not the one in the book.

7 Problem 06.01.07 - Strogatz (Nullclines versus stable manifolds)

Statement for problem 06.01.07

There is a confusing aspect of Example 6.1.1 in the book,¹ dealing with the system

\[
\frac{dx}{dt} = x + e^{-y}, \quad \text{and} \quad \frac{dy}{dt} = -y. \tag{7.1}
\]

The nullcline \( \dot{x} = 0 \) in Figure 6.1.3 has a similar shape and location as the stable manifold of the saddle, shown in Figure 6.1.4. But they are not the same curve! To clarify the relation between the two curves, plot both of them on the same phase portrait. You have two options here. Either:

1. Use a computer to do the plot, generating the stable manifold by numerically solving the equation; or
2. Find an explicit formula for the stable manifold, and then do a sketch of the phase portrait.

\text{Hint. Write the equation for } \frac{dy}{dx} \text{, and then solve it.}

A sketch without an analytical justification is not an acceptable answer!

THE END.