1 Find and classify bifurcations problem #01

Statement: Find and classify bifurcations problem #01

For equation (1.1) below, find the values of $r$ at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points $x^*$ versus $r$.

$$\frac{dx}{dt} = r - \frac{x^2}{1 + x^2}. \quad (1.1)$$

2 Find and classify bifurcations problem #02

Statement: Find and classify bifurcations problem #02

The only extra question that you need to answer is: does the PCS apply across $r = 0$? The others are optional. Note that the problem has three parts, and that in each you have to answer the same set of questions.

Part 1 of 3. For equation (2.1) below, find the values of $r$ at which a bifurcation occurs, and classify them as saddle-node, transcritical, supercritical pitchfork, or subcritical pitchfork. Finally, sketch the bifurcation diagram of fixed points $x^*$ versus $r$.

$$\frac{dx}{dt} = rx - \frac{x^3}{1 + 2x^2 + x^4}. \quad (2.1)$$

Extra questions: Something “strange” happens for $r = 0$ in the bifurcation diagram. Is some bifurcation taking place there? If so, which type? Does the PCS apply across $r = 0$?
Hint 2.1 Look at the equation satisfied by \( y = 1/x \). What happens near \((y, r) = (0, 0)\)?

Remark 2.1 The Principle of Conservation of Stability (PCS) says that: Consider the ode \( \dot{x} = f(x, r) \), where \( f \) is smooth and \( r \) is a parameter. Assign a weight \( w = 1 \) to each stable critical point, a weight \( w = -1 \) to each unstable critical point, and a weight \( w = 0 \) to each semi-stable critical point. Then the sum of the weights (the stability index \( S \)) does not change as \( r \) crosses a bifurcation value.

Part 2 of 3. Consider the equation
\[
\frac{dx}{dt} = rx - \frac{x}{\sqrt{1 + x^2}},
\]
and repeat the analysis in part 1. **Important:** Be careful when doing the transformation to the variable \( y \), as things are not entirely smooth at \( y = 0 \). It follows that what happens near \((y, r) = (0, 0)\) does not fit the “standard” canonical forms studied in the lectures. Nevertheless, you should be able to do it with minimum effort.

Part 3 of 3. Consider the equation
\[
\frac{dx}{dt} = rx - x \text{sech}(x),
\]
and repeat the analysis in part 1. **Note:** the situation near \((y, r) = (0, 0)\) is even less “friendly” than the one in part 2. Yet, it is still tractable if you are careful.

3 Toy model for column buckling

Statement: Toy model for column buckling

Imagine a vertical cylindrical elastic column, on which you push down along its axis by putting a weight on top. If the load is small enough, the column compresses a little, and the elastic response can balance the weight — with the cylinder staying straight. But, if the load is too large, this configuration is not stable, and the column buckles under the weight. This behavior arises because of the interplay of three forces: (i) the load; (ii) the elastic force along the axis of the cylinder; and (iii) the restoring force that is generated when the cylinder bends. When the axial forces are too large, the bending resistance is not enough to keep the straight state stable.

In this exercise we consider a very simple (1-D) toy model, exhibiting the essentials of the behavior described in the prior paragraph. Note though that it is an over-simplified “toy” model, where all the richness of the original setting is gone, and only the column buckling bifurcation remains.

A bead of mass \( m \) (black square) can slide along a rigid horizontal rod (in red). The bead is connected by two equal springs (in blue), with spring constants \( k \), to two supports placed symmetrically a distance \( d \) above and below the rod. A third spring, with spring constant \( s \), pulls the bead towards the middle of the vertical line connecting the supports for the other springs. Everything is frictionless, except for the friction force opposing the motion of the bead along the rod. See the text for further details.

Figure 3.1: Toy model for column buckling.

A sketch depicting the model is shown in figure 3.1. Further assumptions and notation are:
1. Note that the whole device is restricted to a plane, with the bead moving along a line.

2. Let $x$ be the distance, along the rod, of the bead from the vertical line joining the spring supports. Let $x > 0$ if the bead is to the right of the supports and $x < 0$ if to the left. **Note the left-right ($x \rightarrow -x$) symmetry of the set-up.**

3. The two main springs are equal, with a rest lengths $L > 0$ and spring constants $k > 0$. Each generates a force along its axis of magnitude (Hook’s law) $F = k(\ell - L)$, where $\ell$ is the spring length. They push if $\ell < L$, and pull if $\ell > L$.

4. The spring aligned with the rod has zero rest length and a spring constant $s > 0$. This spring generates a restoring force $F = -sx$ along the rod, pulling the bead towards $x = 0$. **Note: a “better” model would have the restoring force provided by a torsion spring located at the hinge between the two springs on the bead. Such a spring would generate a restoring torque proportional to the angle between the two main springs.** However, there is no qualitative difference between these two set-ups — and the one here yields simpler algebra.\(^1\)

5. When the bead slides along the rod, the motion is opposed by a friction force of magnitude $b\dot{x}$, where $b > 0$ is a constant.

6. The distance of the main spring supports from the rod is $d > 0$. Instead of considering the behavior of the system as a function of an applied compression force, we will consider it as a function of the total “imposed” length $2d$ of the “column”.

7. Because the rod is rigid, we need to consider only the horizontal components of the various forces that act on the bead. These are the forces provided by the three springs, and friction along the rod.

**PROBLEM TASKS:**

A. **Derive an ode for the bead position, and write it in appropriate a-dimensional variables.\(^2\)**

B. **Assume that friction is large, so that inertia can be neglected. Exactly which a-dimensional number has to be small for friction to be “large”?**

C. **Analyze the bifurcations that occur for the equation resulting from item B, as the distance $d$ changes (with everything else fixed). What type of bifurcation(s) occur?**

D. **Consider the model that results from neglecting inertia. The equation for this model can, with an appropriate scaling, be written in such a way that it contains a single a-dimensional parameter. Exhibit this form.**

To standardize the notation used in the answers, define

$$a = \frac{L}{d} \quad \text{and} \quad \gamma = \frac{s}{2k}. \quad (3.1)$$

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**Problems motivated by Strogatz’s book**

The problems below are either from Strogatz’s book, or are motivated by problems there. **If different, do the version here, not the one in the book.**

4 Problem 16.09.17 - (exponential to algebraic decay transition)

**Statement for problem 16.09.17**

Consider the equation\(^3\) $\dot{x} = -rx - x^3$, with initial data $x(0) = x_0$. **In the limit $r \downarrow 0$, the solution to this equation transitions from exponential decay as $t \rightarrow \infty$ (when $r > 0$) to algebraic decay when $r = 0$.**

**Why?** Because when $x$ becomes very small, the behavior of the solution is controlled by the term $-rx$ if $r > 0$. On the other hand, when $r = 0$, the equation becomes $\dot{x} = -x^3$, which is known to have algebraic decay.

\(^1\) Both models are over-simplifications of the situation described in the first paragraph of the exercise. There is no point in worrying about getting small details right, when whooping simplifications occur elsewhere.

\(^2\) Suggestion: to a-dimensionalize use $d$ for length and $b/(2k)$ for time.

\(^3\) Note that this equation is one of the two possible normal forms for pitchfork bifurcations.
hand, if \( r = 0 \), the exact solution \( x = x_0/\sqrt{1 + 2x_0^2t} \) decays like \( 1/\sqrt{t} \). However, how do we know that the solutions decay when \( r > 0 \)? Because then \( \dot{x}/x = -(r + x^2) < r \), so that the solution size is less than \( |x_0| e^{-rt} \).

Assume that \( 0 < r \ll 1 \). Then the solution should decay exponentially fast as \( t \to \infty \), yet (because \( r \) is small) the solution should also be “close” to the solution to the problem when the term \(-r x\) is neglected. That is: (4.1).

**How can these two things happen simultaneously?**

(a) Use separation of variables (or any other method) to solve \( \dot{x} = -r x - x^3 \), \( x(0) = x_0 \), \( r > 0 \), analytically. 

Hint: what equation does \( y = 1/x^2 \) satisfy? Once you know \( y \), finding the sign of \( x \) is easy.

(b) For the solution in item a, show that \( x \sim C e^{-rt} \) for \( t \to \infty \), where \( C \) is a constant that you should compute.

(c) For the solution in item a, show that \( x \sim x_\ast \) for \( 0 \leq r t \ll 1 \), where \( x_\ast \) is as in (4.1). Notice that, as \( r \downarrow 0 \), the time interval over which this is valid gets larger and larger.

Hint: The solution in part (a) will involve \( e^{-rt} \). Considers what happens when \( r \) is small and \( 0 \leq t \ll 1/r \), so that \( rt \) is small. Use this to approximate \( e^{-rt} \) in the solution.

(d) To get some intuition on what is going on, plot the exact solution you obtained in item a versus \( x_\ast \) in (4.1) [plot #1]. In addition, plot the exact solution you obtained in item a versus the approximation in item b [plot #2]. Use the same parameter values for both plots, but different time ranges. Suggestion: Use \( x_0 = 1 \) and \( r = 0.01 \). Then, for plot #1 use the range \( 0 < t < 50 \), and for plot #2 the range \( 0 < t < 100 \).

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5 Problem 02.05.06 - Strogatz (The leaky bucket)

**Statement for problem 02.05.06**

(\textit{The leaky bucket}). The following example\(^4\) shows that in some physical situations, non-uniqueness is natural and obvious, not pathological.

Consider a water bucket with a hole in the bottom. If you see a water bucket with a puddle beneath it, can you figure out when the bucket was full? No, of course not! It could have finished emptying a minute ago, ten minutes ago, or whatever. The solution to the corresponding differential equation must be non-unique when integrated backwards in time.

Here is a crude model for the situation. Let \( h(t) = \) height of the water remaining in the bucket at time \( t \); \( a = \) area of the hole; \( A = \) cross-sectional area of the bucket (assumed constant); \( v(t) = \) velocity of the water passing through the hole.

a. Show that \( a v(t) = A \dot{h} \). What physical law are you invoking? \textbf{Warning:} since \( \dot{h} < 0 \), this presumes that we assign a negative value to the velocity \( v \). This is a \textit{weird choice}, implicit in the problem statement, but acceptable.

b. To derive an additional equation, use conservation of energy. First, find the change in potential energy in the system, assuming that the height of the water in the bucket decreases by an amount \( \Delta h \), and that the water has density \( \rho \). Then find the kinetic energy transported out of the bucket by the escaping water. Finally, assuming all the potential energy is converted into kinetic energy, derive the equation \( v^2 = 2gh - g = \text{gravity acceleration} \).

c. Combining a and b, show that \( \ddot{h} = -C \sqrt{\dot{h}} \), where \( C = \frac{a}{A} \sqrt{2g} \).

d. Given \( h(0) = 0 \) (bucket empty at \( t = 0 \)), show that the solution for \( h(t) \) is non-unique \textit{in backwards} time, i.e., for \( t < 0 \).

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\(^5\) Note that, in this problem, evaporation effects are neglected.
The description/derivation above ignores surface tension. Briefly discuss the effect surface tension has on the outcome.

6  Problem 03.02.06 - Strogatz (Eliminate the cubic term)

Statement for problem 03.02.06

Consider the system
\[
\frac{dX}{dt} = RX - X^2 + aX^3 + O(X^4),
\]
where \( R \neq 0 \). We want to find a new variable \( x \) such that the system transforms into
\[
\frac{dx}{dt} = Rx - x^2 + O(x^4).
\]
This would be a big improvement, since the cubic term has been eliminated and the error term has been bumped to fourth order. In fact, the procedure to do this (sketched below) can be generalized to higher orders. This generalization is the subject matter of problem 03.02.07.

Let \( x = X + bX^3 + O(X^4) \), where \( b \) is chosen later to eliminate the cubic term in the differential equation for \( x \). This is called a near-identity transformation, since \( x \) and \( X \) are practically equal: they differ by a cubic term. Now we need to rewrite the system in terms of \( x \); this calculation requires a few steps.

1. Show that the near-identity transformation can be inverted to yield \( X = x + cx^3 + O(x^4) \), and solve for \( c \).
2. Write \( \dot{x} = \dot{X} + 3bX^2\dot{X} + O(X^4) \), and substitute for \( X \) and \( \dot{X} \) on the right hand side, so that everything depends only on \( x \). Multiply the resulting series expansions and collect terms, to obtain \( \dot{x} = Rx - x^2 + kx^3 + O(x^4) \), where \( k \) depends on \( a, b, \) and \( R \).
3. Now the moment of triumph: choose \( b \) so that \( k = 0 \).
4. Is it really necessary to make the assumption that \( R \neq 0 \)? Explain.

7  Problem 03.04.11 - Strogatz (An interesting bifurcation diagram)

Statement for problem 03.04.11

(An interesting bifurcation diagram). Consider the system
\[
\frac{dx}{dt} = rx - \sin(x).
\]
A. For the case \( r = 0 \), find and classify the fixed points, and sketch the vector field.
B. Show that, when \( r > 1 \), there is only one fixed point. What kind of fixed point is it?
C. As \( r \) decreases from \( \infty \) to 0, classify all the bifurcations that occur.
D. For \( 0 < r \ll 1 \), find an approximate formula for the values of \( r \) at which bifurcations occur.
E. Now classify all the bifurcations that occur as \( r \) decreases from 0 to \( -\infty \).
F. Plot the bifurcation diagram for \( -\infty < r < \infty \), and indicate the stability of the various branches of fixed points.

\(^6\) Obviously we are considering here a situation where \( X \) (and \( x \)) is small.
\(^7\) That is, one can successively eliminate all the higher order terms: \( O(x^3), O(x^4), \ldots \), etc.
\(^8\) We have skipped the quadratic term \( X^2 \), because it is not needed — you should check this later.
8  Problem 16.09.20 (growing yeast)

Statement for problem 16.09.20

A (late) medieval\(^9\) baker grows yeast in a vat in his basement. Every day he collects the amount he needs to make bread, which is more or less constant. He is very careful to keep the vat at constant temperature (warm, but not hot), and feeds the yeast regularly with water and flour. An extremely simple model for the yeast is in (8.1).

\[
d\frac{N}{dt} = r N \left(1 - \frac{N}{K}\right) - H. \quad (8.1)
\]

In this model, in the absence harvesting, the yeast the population is assumed to grow logistically. The effects of the harvesting are modeled by the (constant) term \(-H < 0\).

Remark 8.2  The fact that the yeast is collected at a constant rate \(H > 0\), independent of the amount present \(N\), is silly. However, the yeast collection is done by the baker’s apprentice, which is not too bright. He simply goes down to the basement (every day) with the same jar, which he fills to the brim. And, if he happens that there is not enough to fill the jar, that day he takes as much as there is.\(^{10}\)

Questions and tasks.

a. Show that the system can be rewritten in dimensionless form as

\[
\frac{dx}{d\tau} = x (1 - x) - h, \quad (8.2)
\]

for suitable defined dimensionless quantities \(x\), \(\tau\), and \(h\).

b. Plot the bifurcation diagram (as a function of the parameter \(h\)) for equation (8.2), and discuss the various behaviors (in time) that the solutions exhibit in the “physical” regime \((x, h \geq 0)\).

c. Show that a bifurcation occurs at a certain value \(h = h_c\), and classify this bifurcation.

d. Discuss the long-term behavior of the yeast in the vat for \(h < h_c\) and \(h > h_c\), and give the biological interpretation in each case.

e. The fact that the harvesting rate is constant leads to silly behavior of the model solutions: the amount of yeast can become negative! Fix the model so as to incorporate into it the exact behavior by the apprentice described in remark 8.2. Revise your analysis above in view of this change.

Note: a smart apprentice would adjust his collection strategy to depend on \(N\), at the very least. In item e you are not being asked to write a model for an improved strategy. The question is: how do you use (8.1) so that it mimics the actual behavior described in remark 8.2?

Remark 8.3 Details about the continuum limit. In order to arrive at an ode, as in (8.1), in a situation where there is a clear discrete event (collecting the yeast once a day, as in remark 8.2) you need the changes between discrete events to be small, so you can approximate the process by a continuous function. But yeast can grow very fast, as much as duplicate its mass in as little as 90 minutes. So, the stuff above contradicts this, and makes the funny story I was trying to embed the problem into fail.\(^{11}\) Here are a couple of ways the science can be fixed:

1. The Apprentice collects the yeast much more frequently, in smaller batches.

2. The vat is not in as good shape as stated earlier, so the yeast grows a lot slower.

At any rate, even if a late medieval bakery is not the best example giving rise to (8.1), there are many situations where something is being produced by either a biological or a chemical process, and that something is being collected more-or-less continuously at some rate. Equation (8.1) models the simplest set-up of this kind.

THE END.

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\(^9\) Late medieval because the use of yeast to make bread did not become common till the Renaissance, at least in Europe.

\(^{10}\) Next day the baker gets another apprentice.

\(^{11}\) If you wish, the same problem any movie involving science and technology runs into. If you put all the details, the story flops.