Problem Set Number 01, (18.353/12.006/2.050)j
MIT (Fall 2021)
Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)
September 16, 2021

Due Fri September 25, 2021.
Turn it in via the canvas course website.

Contents
1 Get equation from phase line portrait problem #04 1
2 Get equation from phase line portrait problem #05 1
3 Implicit function problem #03 2
   Find the implicit function .................................................. 2
4 Inverse function problem #02 2
   Compute the coefficients of the Taylor expansion for the inverse function .... 2
5 Phase line portrait problem #01 2
   Draw phase line portrait, and indicate critical points and their stability properties ...... 2
6 Problem 02.02.12 - Strogatz (A nonlinear resistor) 3
7 Problem 02.03.3e - Strogatz (Tumor growth) 3
8 Problem 02.03.4t - Strogatz (The Allee effect) 3

1 Get equation from phase line portrait problem #04

Statement: Get equation from phase line portrait problem #04
Consider the ode on the line
\[ \frac{dx}{dt} = f(x), \] (1.1)
where \( f \) is some function which has (at least) one continuous derivative. Assume that (1.1) has exactly two critical points (i.e.: \( x_1 \) and \( x_2 \), with \(-\infty < x_1 < x_2 < \infty\)). Assume also that \( x_1 \) is unstable\(^1\) and that \( x_2 \) is semi-stable.\(^2\) Is this possible? Does a function \( f = f(x) \) yielding this exist?
If the answer is no, prove it.
If the answer is yes, prove it by giving an example.

\[ ^1 \text{A critical point is unstable if the solutions diverge from the critical point on both sides.} \]
\[ ^2 \text{A critical point is semi-stable if the solutions diverge from the critical point on one side, and converge on the other.} \]
where \( f \) is some function which has (at least) one continuous derivative. Assume that (2.1) has exactly two critical points (i.e.: \( x_1 \) and \( x_2 \), with \( -\infty < x_1 < x_2 < \infty \)). Assume also that both critical points are stable. **Is this possible?**

Does a function \( f = f(x) \) yielding this exist?

If the answer is no, prove it.

If the answer is yes, prove it by giving an example.

### 3 Implicit function problem #03

**Statement:** Implicit function problem #03

Consider the following equation

\[
f(x, \lambda) = x + \lambda \sin(x) = 0,
\]

with the particular solution \((x, \lambda) = (0, 0)\). Since \( f_x(0, 0) = 1 \), the implicit function theorem guarantees that:

- there is a neighborhood of \( \lambda = 0 \) where (3.1) has a unique solution, \( x = X(\lambda) \), such that \( X(0) = 0 \).

Find the function \( X(\lambda) \).

### 4 Inverse function problem #02

**Statement:** Inverse function problem #02

Consider the following equation

\[
y = 2x - \frac{1}{\pi}x^3 + \frac{1}{24}x^5 + 0.01x^7 = f(x),
\]

where \( f(0) = 0 \) and \( f'(0) = 2 \neq 0 \). The inverse function theorem guarantees that there is a neighborhood of \( x = 0 \) where \( f \) has a unique inverse, \( x = X(y) \), such that \( X(0) = 0 \). Furthermore, since \( f \) is an analytic function, \( X \) is an analytic function. This means that \( X \) has a Taylor series

\[
X = \sum_{n=0}^{\infty} x_n y^n,
\]

which converges for \(|y|\) small enough. [#1] Find \( x_1, x_3, x_5, \) and \( x_n \) for all even \( n \). Next: [#2] numerically test how good the approximation is. To do so compute \( e_r = |f(X) - y| \), where \( X = \sum_{n=0}^{6} x_n y^n \), and plot \( e_r \) versus \( y \) for \( 0.02 \leq y \leq 1 \). Use a log-log plot. Finally [#3] How does the error behave?

**Hint #1:** Substitute (4.2) into (4.1) and reorganize the result in terms of powers of \( y \). Then make the coefficient of each power of \( y \) vanish. This will give you equations that will allow you to solve for \( x_0 \) first, then \( x_1 \), and so on.

**Hint #2:** There is a special property of the equation that tells you what \( x_n \) is for \( n \) even. **Figure this out first**, it will save you a lot of pointless algebra. This problem involves much less calculation than you might think.

**Hint #3:** What does it mean when you see a straight line in a log-log plot? What does the slope of the plot tell you?

### 5 Phase line portrait problem #01

**Statement:** Phase line portrait problem #01

Consider the following ode on the line

\[
\frac{dx}{dt} = f(x) = \text{sech}(x) - \frac{4}{5}.
\]

Draw its phase line portrait, indicating the critical points, and their stability properties.

---

[#1]: You only need to carry this calculation up to powers \( y^n \) with \( n \leq 5 \).
**Problems motivated by Strogatz’s book**

The problems below are either from Strogatz’s book, or are motivated by problems there. **If different, do the version here, not the one in the book.**

6 **Problem 02.02.12 - Strogatz (A nonlinear resistor)**

Statement for problem 02.02.12

(A nonlinear resistor). Suppose the resistor in Example 2.2.2 (page 20) is replaced by a nonlinear resistor. In other words, this resistor does not have a linear relation between voltage and current. Such a nonlinearity arises in certain solid-state devices. Instead of $I = V/R$, suppose that we have $I = g(V)$, where $g(V)$ has the shape shown in Figure 3 (page 38 of the book).

Redo Example 2.2.2 in this case. Derive the circuit equations, find all the fixed points, and analyze their stability. What qualitative effects does the nonlinearity introduce (if any)?

7 **Problem 02.03.3e - Strogatz (Tumor growth)**

Statement for problem 02.03.3e

(Tumor growth). The growth of cancerous tumors can be modeled by $\dot{N} = -aN \ln(bN) = f(N)$ (Gompertz law), where $N(t)$ is proportional to the number of cells in the tumor, and $a, b > 0$ are parameters.

a. Interpret $a$ and $b$ biologically.

b. Sketch the vector field and then graph $N(t)$ for various initial values.

The predictions of this simple model agree surprisingly well with data on tumor growth, as long as $N$ is not too small. For examples see:


and


Extra questions. (i) What are the zeros of $f$ and what can you say about $f'$ there? (ii) Where is $f$ positive? (iii) Where are the maximums of $f$, and what is the growth rate $\dot{N}/N$ there? (iv) What are the critical points, and what is their stability? (v) What is the behavior of the solutions, as $t \to -\infty$, which approach $N = 0$ as $t \to -\infty$? (vi) Write down exact expressions for all the solutions such that $N > 0$.

8 **Problem 02.03.4t - Strogatz (The Allee effect)**

This problem is motivated by problem 2.3.4 in Strogatz book.
Statement for problem 02.03.4t

The classical view of population dynamics stated that, due to competition for resources, the population growth rate $\dot{N}/N$ ought to steadily decrease with the population density $N$. However, in the 1930 Allee and Bowen\(^4\) were able to demonstrate that goldfish grow more rapidly when there are more individuals within the tank. What happens is that, for many species of organisms, the effective growth rate $\dot{N}/N$ is highest at some intermediate $N$ between $N = 0$ and the saturation density $N = K$. This is called the Allee effect.\(^5\) For example, imagine that it is too hard to find mates when $N$ is very small, and there is too much competition for food and other resources when $N$ is large.

![Graphs of f(N) for No Allee effect, Weak Allee effect, and Strong Allee effect](image)

**Figure 8.1**: (Problem 02.03.4t). The Allee effect, weak and strong.

Mathematically, in terms of the simple model $\dot{N} = f(N)$, we consider three cases (see figure 8.1)

1. **No Allee effect**, where $f'(N)$ decreases with $N$.
2. **Weak Allee effect**, where $f'(N)$ starts positive at $N = 0$, and increases till some critical $N_1$ (an inflection point), and decreases thereafter.
3. **Strong Allee effect**, where $f'(N)$ starts negative at $N = 0$, and increases till some critical $N_1$ (an inflection point) where $f'(N_1) > 0$, and decreases thereafter. We also assume that $f$ is positive near and below $K$ — thus there must be a (single) critical point somewhere in $0 < N < K$.

Note that the mathematical scenarios above are more restrictive than what the “in words” description above allows.

**Show that:**

- **a.** It is possible to have $f = f(N)$ such that $f/N$ decreases with $N$ (thus no Allee effect), but $f'$ is actually increasing somewhere in the interval $0 < K < N$.
- **b.** A weak Allee effect is possible with several inflection points. Here “weak Allee” means: there exists $N_1$ such that $f/N$ increases for $0 < N < N_1$, and decreases for $N_1 < N < K$. Further: $f > 0$ for $0 < N < K$.
- **c.** A strong Allee effect is possible with several inflection points. Here “strong Allee” means: there exists $N_1$ such that $f/N$ increases for $0 < N < N_1$, and decreases for $N_1 < N < K$. Furthermore: $f < 0$ near $N = 0$ and $f > 0$ near $N = K$.

THE END.

---
