Answers to P-Set # 08, (18.353/12.006/2.050)j
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1 Liapunov vs Lyapunov vs Ljapunov . . . Which one?

If you search the web, you will find, mostly Lyapunov being used. However, if you look at dynamical system books (e.g.: see the books in the syllabus), what you find is (mostly) Liapunov. For example:

Liapunov used by Strogatz,

Liapunov used by Wiggins,

Liapunov used by Guckenheimer and Holmes,

Liapunov used by Bergé, Pomeau, and Vidal.

Liapunov used by Drazin,

Ljapunov used by Peitgen, Jürgen, and Saupe.

Lyapunov used by Parker and Chua, *Practical Numerical Algorithms for Chaotic Systems* — this one is not in the syllabus.

Why is this? I think it has to do with the ambiguity in translating from the Cyrillic to the Latin alphabet. Technically, none of the above is correct, because the actual name of the person we are referring too (a.k.a. Alexander Liapunov) was written in Cyrillic; something like **Миха́йлович Ляпуно́в**

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2 Liapunov exponents for 1-D maps #03

2.1 Statement: Liapunov exponents for 1-D maps #03

Task#1 of 3. Compute the Liapunov exponent, and make a plot analog to figure 10.5.2 in Strogatz's book (i.e.: example 10.5.3) for the 1-D maps $x_{n+1} = f(x_n,\, r)$ below. In all cases *justify the selected ranges for x and r.*
Meaning of "justify". Show that if x_n is in the x-region, then so is x_{n+1}. Thus it makes sense to iterate the map.

1. The **Max-μ map** $f(x) = r\,(1 - |2\,x - 1|^{\mu})$, with $\mu = 1.5$. Range $0 \le r \le 1$ and $0 \le x \le 1$. In addition, make plots for the refined regions:
   (1a) $0.7930 \le r \le 0.8170$ and $0 \le x \le 1$.
   (1b) $0.9236 \le r \le 0.9261$ and $0 \le x \le 1$.
2. **Max-μ map** $f(x) = r\,(1 - |2\,x - 1|^{\mu})$, with $\mu = 2.5$. Range $0 \le r \le 1$ and $0 \le x \le 1$. In addition, make plots for the refined regions:
   (2a) $0.925400 \le r \le 0.928400$ and $0 \le x \le 1$.
   (2b) $0.928078 \le r \le 0.928097$ and $0 \le x \le 1$.
   (2c) $0.928095 \le r \le 0.928099$ and $0 \le x \le 1$.

General remarks and background needed for Task#2.
These maps behave in a fashion similar to the Logistic map $f(x) = 4\,r\,x\,(1-x)$, with a series of bifurcations as r grows, at values: $0 < r_1 < r_2 < \cdots < r_\infty < 1$.

At $r = r_1$ the origin becomes unstable, and a new (stable) critical point $0 < x_* < 1$ is born past it.
At $r = r_2$ a (stable) period two solution is born when x_* becomes unstable (flip bifurcation).
At $r = r_3$ a period doubling occurs, followed by further period doublings at each $r = r_n$, $3 < n < \infty$.
At $r = r_\infty$ a transition to chaos occurs.
Beyond $r = r_\infty$ more bifurcations occur, with periodic "windows" (each including within it a period doubling cascade) interspaced with chaotic regions.

In each interval $r_n < r < r_{n+1}$ the (global) attractor is a periodic solution [1] of period 2^{n-1}. At a particular point $r_n < r = s_n < r_{n+1}$ in each interval the attractor is **super-stable** — meaning that perturbations evolve according to $\epsilon_{n+1} = O(\epsilon_n^2)$. Furthermore:

The r_n converge geometrically to r_∞: $R_n = (r_{n-1} - r_{n-2})/(r_n - r_{n-1})$ is approximately constant.
The s_n converge geometrically to r_∞: $S_n = (s_{n-1} - s_{n-2})/(s_n - s_{n-1})$ is approximately constant.
The Feigenbaum number

$$\delta_{\text{map}} = \lim_{n\to\infty} R_n = \lim_{n\to\infty} S_n \tag{2.1}$$

characterizes the convergence rate; i.e.: $r_\infty - r_n$ and $r_\infty - s_n$ behave like $(\delta_{\text{map}})^{-n}$ for $n \gg 1$.

Note that *the r_n are difficult to compute accurately,* because they correspond to neutrally stable solutions. On the other hand, *the s_n are relatively easy to compute accurately,* since they correspond to super-stable solutions. Hence the second equality in (2.1) provides the *better way to compute* δ_{map}.

How does this all relate to the Liapunov exponent $\lambda_e(r)$?

Chaos yields $\lambda_e > 0$.
Period doubling: r_n is a local maximum of λ_e, where $\lambda_e = 0$. That is: $\lambda_e(r_n) = 0$ and $\lambda_e < 0$ on each side of r_n.
Super-stable attractors yield $\lambda_e = -\infty$. However, *in a numerical calculation this will, generally, not be true.* Instead, downward (negative) spikes in the plot of λ_e versus r (centered at $r \approx s_n$) are seen.

Task#2 of 3. In both cases above for the **Max-μ map** ($\mu = 1.5$ and $\mu = 2.5$), calculate a few s_n (say, for $1 \le n \le 4$) and use these values to get an estimate for $\delta_{\text{Max-1.5}}$ and $\delta_{\text{Max-2.5}}$. Note that 3 significant digits for s_n, $2 \le n \le 4$, will allow you to get S_4 with about 2 significant digits.

Task#3 of 3 (optional). The plot for **(2c)** shows the transition from period doubling to the "beyond r_∞" chaos region — as seen from point of view of the Liapunov exponent. **What does the plot for (1b) show?**

Hints, process, and remarks.

[1] For $n = 1$ the attractor is actually critical point — i.e.: period = 1.

h1. The process to compute the Liapunov exponent λ_e is explained in example 10.5.3 of the book by Strogatz. Specifically: for a given value of r, select a random $0 < x_1 < 1$. Then iterate the map, $x_{n+1} = f(x_n,\, r)$, for $1 \leq n \leq n_b$, where $n_b \gg 1$ (say: a few thousand times) — this so the attractor is reached. Next continue the map iteration, and start the computation of λ_e, as follows: (i) Define $\lambda_1 = 0$. (ii) Let $\lambda_{m+1} = \lambda_m + \frac{1}{n_p}\,\log(|f'(x_n,\, r)|)$, for $1 \leq m \leq n_p$, where $n = n_b + m$, $n_p \gg 1$ (a few thousands, at least), and $f' = \mathrm{d}f/\mathrm{d}x$. (iii) Then $\lambda_e \approx \lambda_{n_p+1}$.

This has to be done for each value of r in a grid within the desired range $r_1 \leq r \leq r_2$. For example, an equi-spaced uniform grid, with separation $\Delta r = (r_2 - r_1)/N$.

h2. If you do this with MatLab, it is important that you "vectorize" the calculation (for speed). Place all the values of r in an array $\vec{r}$, with the corresponding values for the iterates (and Liapunov exponent approximation) in arrays $\vec{x}_n$ and $\vec{\lambda}_m$, and compute everything simultaneously — note that, at any stage in the iterations, you need to keep (at most) values for two n and two m. Finally: use the MatLab command "print -dpng FigureName" to save the figure as a small png file — this is more reliable than trying to use the GUI in the figure window.

h3. Calculation of the s_n. Since the S_n involve differences of s_n, significant digits are lost in the calculation. To get s_n accurate enough, you will need a fine grid in r (small Δr). However, even with a laptop, a "vectorized" MatLab calculation with $N \sim 10^5$ values of r will take, at most, a few minutes. The main limitation in (2.1) is that, in principle, you need "large" values of n to accurately compute δ_{map} — but you are not asked to do this.

Finally, note that you can get the required values of the s_n graphically, by using "zoom" and the "data cursor" in the MatLab figure window with the plot of λ_e versus r.

2.2 Answer: Liapunov exponents for 1-D maps #03

The answer to the three tasks follows below.

Task#1. We begin by the justification of ranges. In both cases $f \geq 0$ for $0 \leq x \leq 1$, with minimum value $f = 0$ at $x = 0,\, 1$, and maximum $f = r$ at $x = 0.5$. Thus f maps the unit interval onto itself, as long as $0 \leq r \leq 1$. The required plots are in figures 2.1–2.3, and the left panel in figure 2.4.

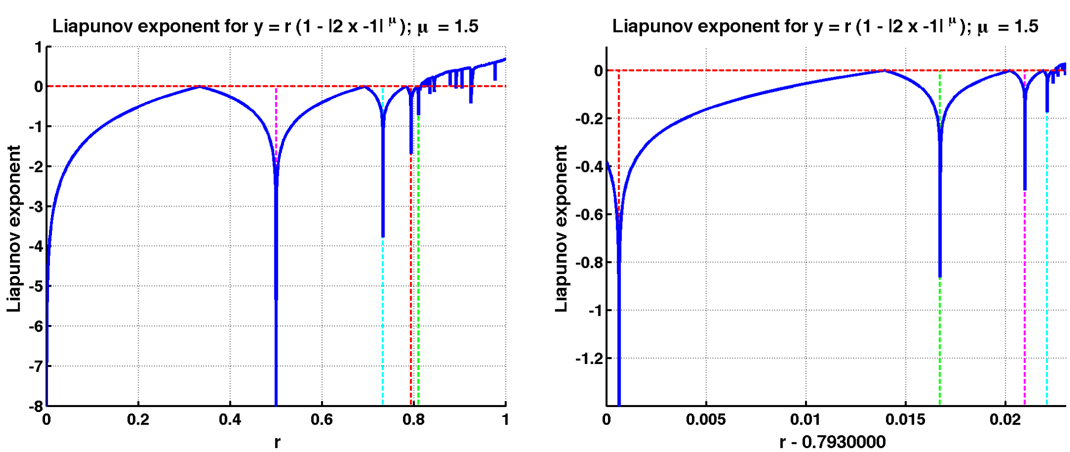

Figure 2.1: Liapunov exp. Max-1.5. s_1 through s_6 marked (cyclic) in magenta, cyan, red, green, . . . For more details **see text.**

The panels in figure 2.1 correspond to the Max-μ map with $\mu = 1.5$. The left panel shows the Liapunov exponent for $0 \leq r \leq 1$, while the right panel shows the Liapunov exponent for $0.7930 \leq r \leq 0.8170$. Note that in the right panel the r-coordinate is shifted by 0.793 — this is because the interval is small, and proper labels do not fit otherwise. Finally, the values $r = s_n$ (corresponding to super-stable attractors) are indicated by color coded vertical lines. The color codes cycle through magenta, cyan, red, and green (starting with magenta for s_1).

Approximate values for the s_n are: $s_2 \approx 0.733$, $s_3 \approx 0.794$, and $s_4 \approx 0.810$,
while $s_1 = 0.5$ (exact). Then (2.1) yields

$$S_4 \approx 3.8. \qquad (2.2)$$

More accurate values are: $s_2 \approx 0.73279$, $s_3 \approx 0.79363$, $s_4 \approx 0.80972$, $s_5 \approx 0.81396$,
and $s_6 \approx 0.81508$. These then lead to:

$$S_3 \approx 3.826,\quad S_4 \approx 3.781,\quad S_5 \approx 3.792,\quad \text{and} \quad S_6 \approx 3.798. \qquad (2.3)$$

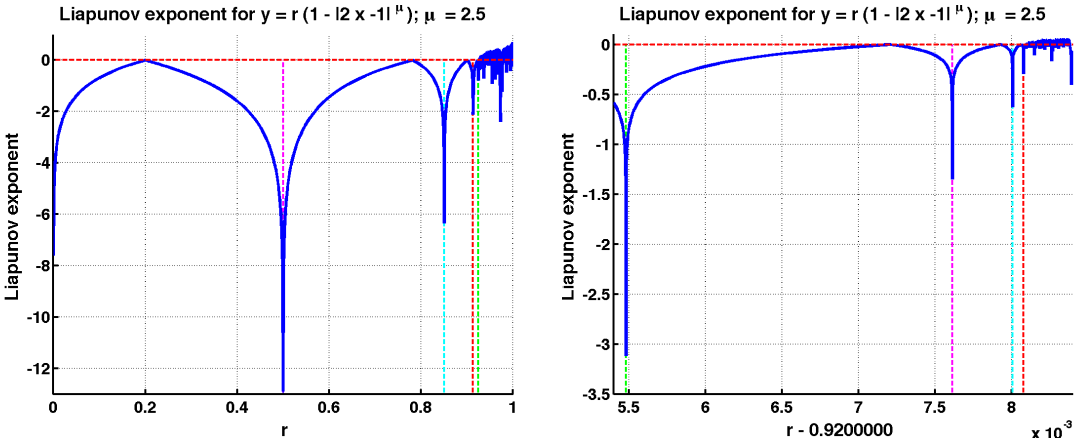

Figure 2.2: Liapunov exp. for Max-2.5. s_1 through s_7 marked (cyclic) in magenta, cyan, red, green, . . . For more details **see text.**

The panels in figure 2.2 correspond to the Max-μ map with $\boldsymbol{\mu = 2.5}$. The left panel shows the Liapunov exponent for $\mathbf{0 \leq r \leq 1}$, while the right panel shows the Liapunov exponent for $\mathbf{0.925400 \leq r \leq 0.928400}$. Note that in the right panel the r-coordinate is shifted by $\mathbf{0.92}$ — this is because the interval is small, and proper labels do not fit otherwise. Finally, the values $\boldsymbol{r = s_n}$ (corresponding to super-stable attractors) are indicated by color coded vertical lines. The color codes cycle through magenta, cyan, red, and green (starting with magenta for s_1).

Approximate values for the $\boldsymbol{s_n}$ are: $\boldsymbol{s_2 \approx 0.851}$, $\boldsymbol{s_3 \approx 0.914}$, and $\boldsymbol{s_4 \approx 0.925}$,
while $\boldsymbol{s_1 = 0.5}$ (exact). Then (2.1) yields

$$\boldsymbol{S_4 \approx 5.7}. \qquad (2.4)$$

More accurate values† are: $\boldsymbol{s_2 \approx 0.85080}$, $\boldsymbol{s_3 \approx 0.91394}$, $\boldsymbol{s_4 \approx 0.925481}$,
$\boldsymbol{s_5 \approx 0.927613}$, $\boldsymbol{s_6 \approx 0.92800684}$, $\boldsymbol{s_7 \approx 0.92807958}$, $\boldsymbol{s_8 \approx 0.928093018}$,
$\boldsymbol{s_9 \approx 0.928095500}$, $\boldsymbol{s_{10} \approx 0.928095959}$, etc.

These then lead to:

$$\boldsymbol{S_3 \approx 5.556}, \ \boldsymbol{S_4 \approx 5.471}, \ \boldsymbol{S_5 \approx 5.414}, \ \boldsymbol{S_6 \approx 5.414}, \ \boldsymbol{S_7 \approx 5.413}, \ \boldsymbol{S_8 \approx 5.413}, \qquad (2.5)$$

$\boldsymbol{S_9 \approx 5.41}$, $\boldsymbol{S_{10} \approx 5.41}$, etc.

† Not sure of how much trust I have on the last digit in s_8 through s_{10}.

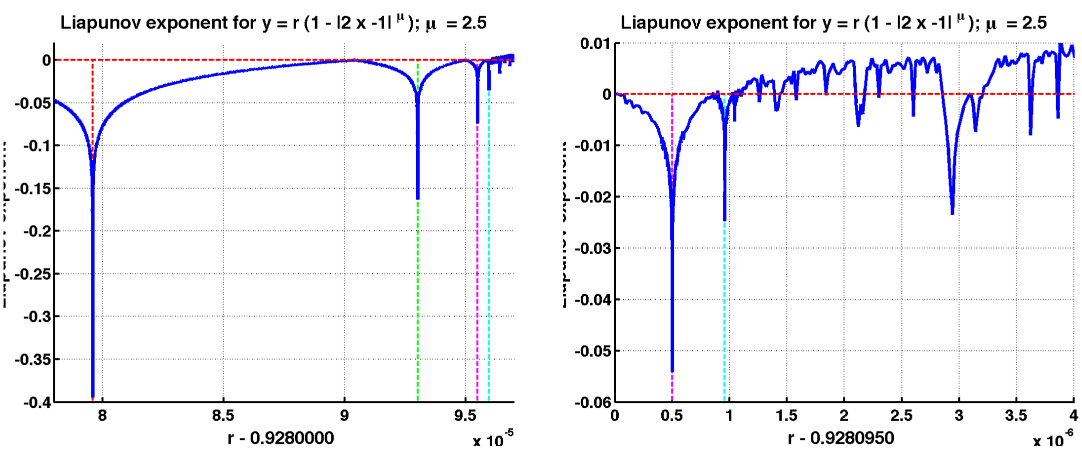

Figure 2.3: Liapunov exp. for Max-2.5. s_7 through s_{10} marked in red, green, magenta, and cyan. For more details **see text.**

The panels in figure 2.3 correspond to the Max-μ map with $\boldsymbol{\mu = 2.5}$. The left panel shows the Liapunov exponent for $\mathbf{0.928078 \leq r \leq 0.928097}$, while the right panel shows the Liapunov exponent for $\mathbf{0.928095 \leq r \leq 0.928099}$. Note that in the left/right panel the r-coordinates are shifted by $\mathbf{0.928/0.928095}$ — this is because the interval is small, and proper labels do not fit otherwise. As before, the values $\boldsymbol{r = s_n}$ (corresponding to super-stable attractors) are indicated by color coded vertical lines. The color codes cycle through red, green, magenta, and cyan (starting with red for s_7). Finally, the **right panel shows the region near r_∞,** with period doubling on the left and chaos on the right. *Within the chaotic region we can see a "large" periodic window, and a few smaller ones* — these are characterized by regions where the Liapunov exponent drops below zero.

Remark. The Liapunov exponent is negative within "periodic windows" (where the attractor is a periodic orbit), and reaches $-\infty$ at the super-stable periodic orbits within each window (lack of numerical resolution cuts these downward "spikes" to a finite value). *In chaotic regions the Liapunov exponent is positive.* Further: *the chaotic regions contain a fractal structure of periodic windows* (hard to resolve beyond the larger ones). These features are evident in the plots of Liapunov exponents shown in this answer. ♣

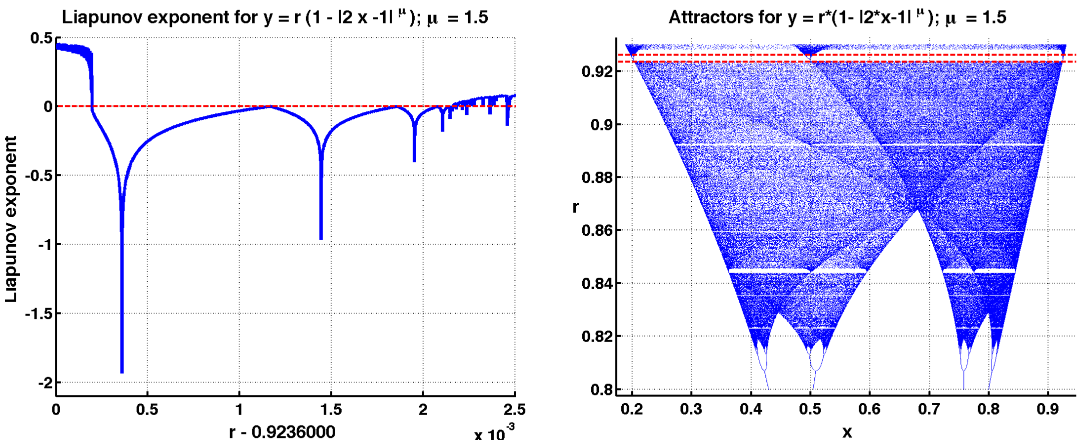

Figure 2.4: Max-1.5 map. Left: Liapunov exponent for the region (1b). Right: partial orbit diagram. For more details **see text.**

The panels in figure 2.4 correspond to the Max-μ map with $\boldsymbol{\mu = 1.5}$. The left panel shows the Liapunov exponent for $\mathbf{0.9236} \leq \boldsymbol{r} \leq \mathbf{0.9261}$. Note that the r-coordinate in this panel is shifted by $\mathbf{0.9236}$ — this is because the interval is small, and proper labels do not fit otherwise. The right panel shows a partial orbit diagram for the map, starting from a period 4 attractor ($\boldsymbol{r = 0.8}$) to a value deep in the chaos region ($\boldsymbol{r = 0.93}$). **In this right panel, the values of r corresponding to the left panel are shown bracketed by two dashed red lines.**

Task#2. This task is done in (2.2–2.3) for $\boldsymbol{\mu = 1.5}$, and in (2.4–2.5) for $\boldsymbol{\mu = 2.5}$. This results in the estimates

$$\delta_{\text{Max-1.5}} \approx \mathbf{3.8} \quad \text{and} \quad \delta_{\text{Max-2.5}} \approx \mathbf{5.7} \qquad (2.6)$$

for the Feigenbaum number.[2] Note that **these values differ from the universal Feigenbaum constant $\delta_2 = 4.6692\ldots$,** but there is nothing wrong with them. Further **this does not contradict the universality of δ_2.** The reason is that $\boldsymbol{\delta_2}$ is universal for unimodal maps where the maximum is quadratic, which is not the case here (where the maximum is controlled by $\mu \neq 2$). In fact, it can be checked that $\boldsymbol{\delta_{\text{Max-}\mu}}$ grows[‡] with $\boldsymbol{\mu}$ (e.g.: $\boldsymbol{\delta_{\text{Max-4}} \approx 7.2}$).

Task#3. The plot for **(1b)** is shown on the left panel of figure 2.4. What we see there is a "periodic window" embedded within the chaotic region — even a period doubling cascade can be clearly seen. The right panel of figure 2.4 shows that **this is, in fact, the period three window.**

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3 Nonlinear stability of a discrete map, and flip bifurcation

3.1 Statement: Nonlinear stability of a discrete map, and flip bifurcation

Consider a 1-D map, $\boldsymbol{x_{n+1} = f(x_n)}$, where $\boldsymbol{f}$ is smooth. Assume a fixed point $\boldsymbol{x_f = f(x_f)}$, where $\boldsymbol{f'(x_f) = -1}$ — hence linearization does not determine the stability of $\boldsymbol{x_f}$. Without loss of generality, assume $\boldsymbol{x_* = 0}$, and write

$$f(x) = -x + a\,x^2 + b\,x^3 + O(x^4), \qquad (3.1)$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are constants. These are **your tasks:**

[2] Note that (2.5) yields the more accurate estimate $\delta_{\text{Max-2.5}} \approx \mathbf{5.413}$.

t1. **Find condition on a and b** that determines wether $x = 0$ is a stable or unstable fixed point. *Hint:*

t1.a The condition looks like: stability if $h(a, b) > 0$, and instability if $h(a, b) < 0$, for some function h.

t1.b Consider what happens upon iterating $g(x) = f(f(x))$, which you can ascertain by expanding g to $O(x^4)$, using (3.1). Then note: if $x_{2n+2} = g(x_{2n})$ decays/grows, then so does x_{2n+3}, because f is continuous.

t2. **Answer this question:** *why do you have to expand g up to $O(x^4)$, in item* t1.b*, to determine stability?* Note that here I expect the mathematical/technical reason for this.

t3. Let a and b in (3.1) be such that $x = 0$ is stable, i.e.: $h(a, b) > 0$, and take a map F such that

$$F(x) = -(1+\delta)\,x + a\,x^2 + b\,x^3 + O(x^4), \tag{3.2}$$

where $0 < \delta \ll 1$. Then x is a linearly unstable fixed point, and a **period two (stable) solution**

$$x^*_{n+2} = x^*_n, \quad x^*_{n+1} = F(x^*_n), \tag{3.3}$$

appears,‡ where x^*_n has size $O(\sqrt{\delta})$.
This is called a **supercritical (or soft) flip bifurcation.**

‡ Argument: the same we made to explain the scaling behind supercritical pitchfork and Hopf bifurcations. The new solution appears as a balance between the destabilizing linearity, and the stabilizing nonlinearity.

Your task. Pick an example F where this happens, with $a \neq 0 \neq b$, and show a numerically computed picture of cobwebs† converging to the period two stable solution.

† Use two cobwebs (with different colors), one converging from "inside" and the other from "outside".

I suggest that you write a "generic" program for $F(x) = -(1+\delta)\,x + a\,x^2 + b\,x^3$ and initial data x_0, and then play with the parameters till you get a pretty picture. Further: choose your colors well; e.g.: yellow on a white background is a bad idea! Note: something like $1 < a < 2$, $b \sim -2/3$, and $\delta \sim 0.3^2$, worked for me.

3.2 Answer: Nonlinear stability of a discrete map, and flip bifurcation

The answers to the tasks are below.

t1. We have: $g(x) = -(-x + a\,x^2 + b\,x^3) + a\,(-x + a\,x^2)^2 - b\,x^3 + O(x^4) = x - 2\,(a^2 + b)\,x^3 + O(x^4)$, where we note that **the quadratic terms cancel!**
This can be written as

$$g(x) = \Big(1 - 2\,(a^2+b)\,x^2 + O(x^3)\Big)\,x = \lambda(x)\,x, \tag{3.4}$$

from which it is clear that:

$$a^2 + b > 0 \Rightarrow 0 < \lambda < 1 \quad \text{for } x \text{ small. Therefore: } x = 0 \textbf{ is stable.} \tag{3.5}$$

$$a^2 + b < 0 \Rightarrow 1 < \lambda \quad \text{for } x \text{ small. Therefore: } x = 0 \textbf{ is unstable.} \tag{3.6}$$

t2. We need to compute up to $O(x^4)$ because, as shown in item **t1**, the $O(x^2)$ vanish, and stability is decided by the $O(x^3)$ terms.

t3. See figure 3.1

THE END.

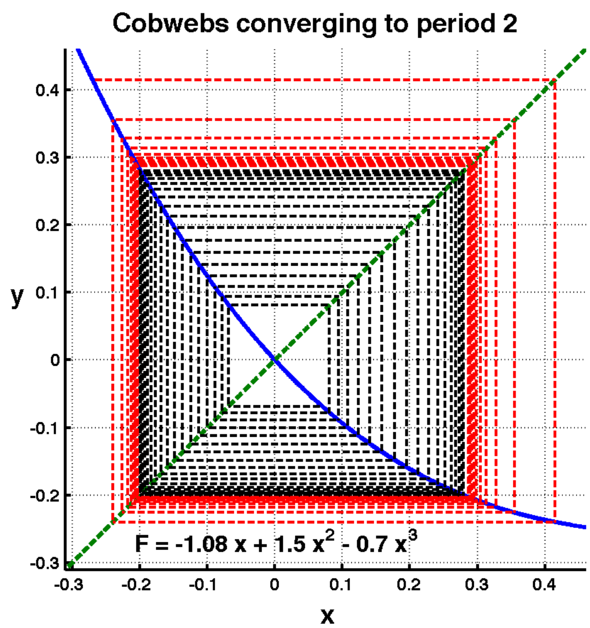

Figure 3.1: Cobwebs converging to period two.
Nonlinear stability of a discrete map, and flip bifurcation. The picture on the left shows two cobwebs, converging towards a stable period two solution (after a supercritical flip bifurcation),
for the map $F(x) = -1.08\,x + 1.5\,x^2 - 0.7\,x^3$.
Note that the
amplitude of the period two solution is $O(\sqrt{\delta})$, as expected (here $\sqrt{\delta} \approx 0.283$).
The function F is plotted in blue, while the green dotted line corresponds to $y = x$.