Answers to P-Set # 07, (18.353/12.006/2.050)j
MIT (Fall 2024)

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Part I

Regular problem set

1 Homoclinic bifurcation with a computer #01

1.1 Statement: Homoclinic bifurcation with a computer #01

Consider the system $$\dot{x} = \mu\, x + y - (2\,\mu + 1)\, x^2 \quad \text{and} \quad \dot{y} = -x + \mu\, y + (2 - \mu)\, x^2, \tag{1.1}$$
where μ is a parameter. For this system:

(1) Find and analyze/classify the critical points.

(2) Using a computer, show that a supercritical Hopf bifurcation happens at $\mu = 0$.

(3) Using a computer, show that an homoclinic bifurcation happens for $0 < \mu = \mu_c$, where μ_c is very small.

You can get a rough idea of "how small is small" by recalling that the "radius" of the Hopf bifurcation limit cycle grows like the square root of the parameter deviation from the onset. Hence, in this case it should be $C\,\sqrt{\mu_c} \approx$ distance from the origin to the saddle, where C is a constant. Of course, you do not know C, so this estimate can be off by some factor, probably less than 10. **Do this estimate before doing (3).**

(4) Find the approximate value of μ_c.

(5) Illustrate your results with phase plane plots in the region $-0.4 < x,\ y < 0.6$.

Hint: The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence *check the behavior of the orbits near the critical point for $\mu = 0$, to see if the nonlinear terms stabilize or de-stabilize.*

Suggestions for computation. Use **PHPLdemoB2**, which allows for easy parameter searches. To get precisely controlled plots, you can use **PHPLplot** or **PHPLplot_v2**, which are a bit less user friendly, but allow precise control. Note that, for μ small, the radial dynamics near the origin is slow, thus you may need "large" integration times ... **do not use the default integration times and tolerances in the scripts.**

1.2 Answer: Homoclinic bifurcation using a computer #01

(1) The critical points are:

- **1a. The origin,** with linearization $\delta\dot{x} = \mu\,\delta x + \delta y$ and $\delta\dot{y} = -\delta x + \mu\,\delta y$.
  This yields: **stable spiral point for $\mu < 0$** and **unstable spiral point for $\mu > 0$.**
- **1b. $(x,\ y) = (1/2,\ 1/4)$,** with linearization $\delta\dot{x} = -(\mu + 1)\,\delta x + \delta y$ and $\delta\dot{y} = (1 - \mu)\,\delta x + \mu\,\delta y$.
  This yields a **saddle for all μ,** with eigenvalues $\lambda = -\frac{1}{2} \pm \sqrt{\frac{5}{4} + \mu^2}$.

Figures 1.1 and 1.2 confirm **1a.** and **1b**

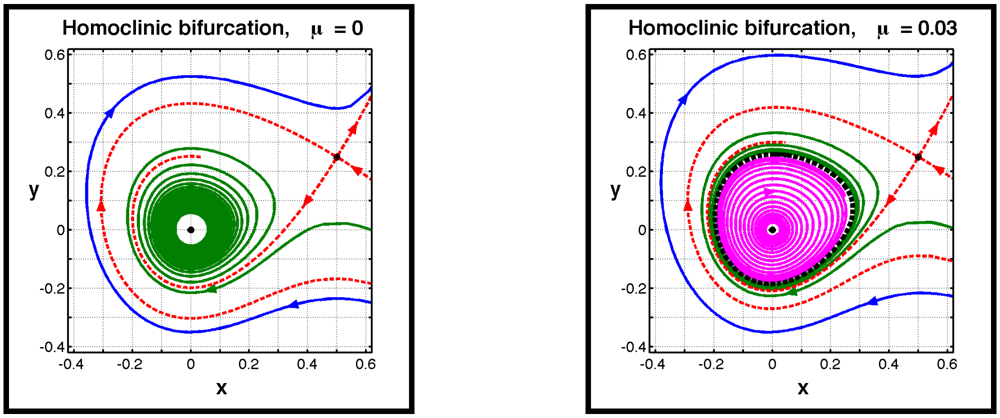

Figure 1.1: Homoclinic bifurcation with a computer #01, for the system in (1.1) — see item **(2)**.

(2) The Hopf bifurcation, figure 1.1. The **left panel** shows the system's phase portrait for $\boldsymbol{\mu = 0}$, with the saddle stable/unstable trajectories in dashed red, and two typical orbits in blue and green. The green trajectory approaches the origin as $\boldsymbol{t \to \infty}$, albeit **very slowly** (so does one of the unstable trajectories from the saddle). We conclude that **the nonlinearity is stabilizing, resulting in a supercritical Hopf bifurcation** — **however: see § 1.2.1.** The **right panel** shows the limit cycle for $\boldsymbol{\mu = 0.03}$ (dashed black line), with the saddle stable/unstable trajectories in dashed red, and three typical orbits in blue, green, and magenta — the last two approach the limit cycle as $\boldsymbol{t \to \infty}$.

Next we do a **gross estimate for $\boldsymbol{\mu_c}$.** The distance from the origin to the saddle is $\boldsymbol{d = \sqrt{5/16}}$, hence $\boldsymbol{\mu_c \approx (5/16)/C^2 \approx 0.31/C^2}$ — as we will see, $\boldsymbol{C \approx 2}$ gives the correct answer.

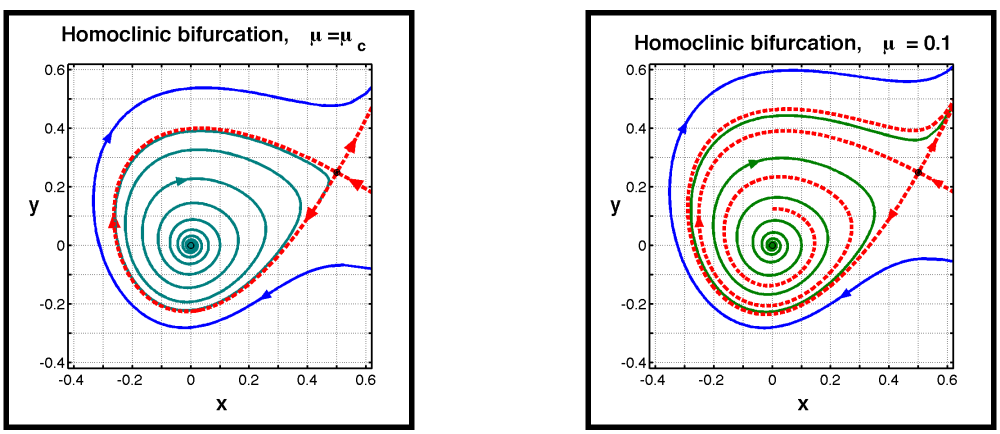

Figure 1.2: Homoclinic bifurcation with a computer #01, for the system in (1.1) — see item **(3)**.

(3) The homoclinic bifurcation, figure 1.2. The **left panel** shows the system's phase portrait for $\boldsymbol{\mu = \mu_c}$, where $\boldsymbol{0.0767 < \mu < 0.0768}$. Here the saddle stable/unstable trajectories are shown in dashed red, and two typical orbits in blue and green. This is the precise **value of $\boldsymbol{\mu}$ at which the limit cycle vanishes, upon colliding with the saddle, and becoming an homoclinic connection.** The orbits inside the cycle graph, in green, spiral out from the origin towards the homoclinic connection. The **right panel** shows the phase plane portrait for $\boldsymbol{\mu = 0.1 > \mu_c}$, with (again) the saddle stable/unstable trajectories in dashed red, two typical orbits in blue and green, and **no limit cycle.**

(4) The critical value is $\boldsymbol{\mu_c = 0.0767\ldots}$ To obtain this value I computed the stable/unstable manifolds for the saddle (in practice: compute trajectories starting from a point very close to the saddle), and vary $\boldsymbol{\mu}$ till an homoclinic orbit is obtained.

(5) Phase plane plots. See figures 1.1 and 1.2.

1.2.1 Analysis for the Hopf bifurcation; a proof that the nonlinearity stabilizes

The computer argument in **(2)***, showing that the nonlinearity stabilizes, has a* **weakness:** The approach of the green trajectory to the origin (left panel in figure 1.1) gets ever slower as time increases, and it requires a bit of a leap of faith to claim that the trajectory approaches the origin as $t \to \infty$. *Here we prove that this is correct, using a Liapunov function.* The $\mu = 0$ system is

$$\dot{x} = y - x^2 \quad \text{and} \quad \dot{y} = -x + 2\,x^2. \tag{1.2}$$

We will show that the origin is (nonlinear) stable spiral for this system (which then implies that a supercritical Hopf bifurcation occurs for $\mu = 0$). To do so, first we eliminate the variable y by taking the time derivative of the first equation, and substituting $\dot{y}$ from the second. Hence

$$\ddot{x} + 2\,x\,\dot{x} + x - 2\,x^2 = 0. \tag{1.3}$$

This equation is fully equivalent to (1.2) via the substitution $y = \dot{x} + x^2$. It can also be written in the form

$$\dot{x} = v \quad \text{and} \quad \dot{v} = -(1 + 2\,v)\,x + 2\,x^2. \tag{1.4}$$

Now, define (see "motivation" below)

$$L = \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,v - \tfrac{1}{4}\ln(1 + 2\,v) - \tfrac{2}{3}\,x^3. \tag{1.5}$$

Then it is straightforward to see that

$$\tfrac{dL}{dt} = -4\,x^2\,v^2/(1 + 2\,v). \tag{1.6}$$

This is **precisely what we were looking for.** Near the origin, $\dot{L} < 0$, except along the coordinate lines $x \equiv 0$ and $v \equiv 0$, where $\dot{L} = 0$. However, it is easy to check that non-trivial orbits cross the coordinate axis near the origin.[1] Hence L is a *strictly decreasing function of time along non-trivial trajectories near the origin.* Further

$$L = \tfrac{1}{2}\,(x^2 + v^2) + \text{higher order terms}, \tag{1.7}$$

so that the origin is a local minimum for L. In addition, it is easy to see that $\dot{\theta} = -1 + O((x^2 + v^2)^{1/2})$, where θ is the polar angle in the $(x,\, v)$ plane. **Hence the origin is a (non-linear) stable spiral point.** ♣

Equation (1.6) "explains" why the spiraling down to the origin of the green orbit in figure 1.1 is so slow. If $r = \sqrt{x^2 + v^2}$, then for a linear spiral $\dot{r} = O(r)$, while (1.6) shows that here $\dot{r} = O(r^3)$; massively slower.

Motivation. Equation (1.5) requires explanation; it is not the kind of thing one just guesses. Here it goes

- **(a)** The reduction from (1.2) to (1.3) is because (1.3) is more intuitive: an oscillator in a **potential** $V = \frac{1}{2}\,x^2 - \frac{2}{3}\,x^3$, with a typical non-linear driven-damping term.
- **(b)** Upon eliminating the non-linear term $2\,x^2$ in the "spring force", we can write $dv/dx = -(1 + 2\,v)\,x/v$, using (1.4). This equation is separable, and can be integrated. The constant of integration then yields an integral of motion
  $$E = \tfrac{1}{2}\,x^2 + \tfrac{1}{2}\,v - \tfrac{1}{4}\ln(1 + 2\,v).$$
  Hence $\ddot{x} + 2\,x\,\dot{x} + x = 0$ **is a conservative system!**
  It is then natural to ask: What happens if the "missing" piece of the potential V, i.e.: $-\frac{2}{3}\,x^3$, is added to E? The result is L, a Liapunov function.

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2 Newton's method in the complex plane #02

2.1 Statement: Newton's method in the complex plane #02

Suppose that you want to solve an equation, $g(x) = 0$. Then you can use *Newton's method*, which is as follows: Assume that you have a "reasonable" guess, x_0, for the value of a root. Then the sequence $x_{n+1} = f(x_n)$, $n \geq 0$, where

$$f(x) = x - \frac{g(x)}{g'(x)}, \tag{2.1}$$

converges (very fast) to the root.

[1] For example, if $x = 0$ and $v \neq 0$, $\dot{x} = v \neq 0$.

Remark 2.1 (The idea). Assume an approximate solution $g(x_a) \approx 0$. Then write $x_b = x_a + \delta x$ to improve it, where δx is small. Then $0 = g(x_a + \delta x) \approx g(x_a) + g'(x_a)\,\delta x \;\Rightarrow\; \delta\,x \approx -\frac{g(x_a)}{g'(x_a)}$, and (2.1) follows.
Of course, if x_0 is not close to a root, the method may not converge. Even if it converges, it may converge to a root far away from x_0, not necessarily the closest root. In this problem **we investigate the behavior of Newton's method in the complex plane, for arbitrary starting points.** ♣

Consider iterations of the map generated by Newton's method for the roots of $z^4 - 1 = 0$. i.e.:

$$z_{n+1} = f(z_n) = \left(\frac{3}{4} + \frac{1}{4\,z_n^4}\right) z_n, \quad n \geq 0, \qquad (2.2)$$

where $0 < |z_0| < \infty$ is arbitrary, and the z_n are **complex numbers.**
Note that

$$\zeta_1 = 1, \quad \zeta_2 = e^{i\,\pi/2} = i, \quad \zeta_3 = e^{i\,\pi} = -1, \text{ and } \zeta_4 = e^{i\,3\,\pi/4} = -i, \qquad (2.3)$$

are the roots of $z^4 = 1$.

Your tasks: Write a computer program to calculate the orbits $\{z_n\}_{n=0}^{\infty}$. Then, for every initial point z_0, draw a colored dot at the position of z_0, where **the colors are picked as follows:**

$$z_n \to \zeta_1, \textbf{ green.} \quad z_n \to \zeta_2, \textbf{ red.} \quad z_n \to \zeta_3, \textbf{ blue.} \quad z_n \to \zeta_4, \textbf{ yellow.} \quad \textbf{No convergence, black.} \qquad (2.4)$$

What do you see? Do blow ups (see item h6 below) of the limit regions between zones.

Hints, practical numerical considerations, etc.

h1. Divide the region [I *strongly* suggest the square $-2 \leq \mathrm{Re}(z_0),\ \mathrm{Im}(z_0) \leq 2$] where the initial data z_0 will be picked into pixels, then pick a z_0 at the center of each pixel, and color the pixel according to (2.4).

h2. If you use MatLab, **do not plot points.** As suggested in item **h1**, plot pixels — use the command image(x, y, C) to plot, where: $x = \mathrm{Re}(z_0)$, $y = \mathrm{Im}(z_0)$, and C encodes the color. **Why?** Because using points leaves a lot of unpainted space in the figure, and **gives huge file sizes** if you use enough pixels to get a good picture.

h3. Deciding convergence. Deciding that the sequence converges is easy: once z_n gets "close enough" to one of the roots, then the very design of Newton's method guarantees convergence. Thus, given a z_0, compute z_N for some large N, and check if $|z_N - \zeta_j| < \delta$ for one of the roots and some "small" tolerance δ — which does not have to be very small, but pick $\delta = 10^{-5}$ for extra caution. If this criteria is not satisfied for any of the roots, then classify the sequence starting at z_0 as "non-convergent".

You can get reasonable pictures with $N = 50$ iterations on a 150×150 grid — a larger N is needed when refining near the boundary between zones. For the answer I used a 500×500 grid and $N = 100$ iterations — which I increased to $N = 150,\ 200,\ 250$ for the blow ups of details (likely over-kill).

h4. Compute in parallel. If you use MatLab, make sure to do all the sequences (one for each pixel) in parallel, using vector/matrix operations. This is much, much, faster than a "for loop".

h5. Avoid division by zero. Note that (2.2) ceases to make sense if $z_n = 0$ — classify this as non-convergence. This can cause a problem if you are computing all the sequences in parallel, because this requires all of them to be computed from z_0 to z_N. One way to get around this (in MatLab) is as follows: Place all the iterates in a complex matrix **Zn**, where the entry $(p,\ q)$ corresponds to z_n for the sequence starting in the $(p,\ q)$ pixel. Then, before computing the next iterate, execute: **Zn = Zn + del*(Zn == 0)**, where **del = 1e-20**.† After this sequences with $z_n = 0$ will produce a very large z_{n+1}, which is guaranteed not return to the vicinity of the roots ζ_j for many iterations (more than 300), resulting in "effective" non-convergence.‡

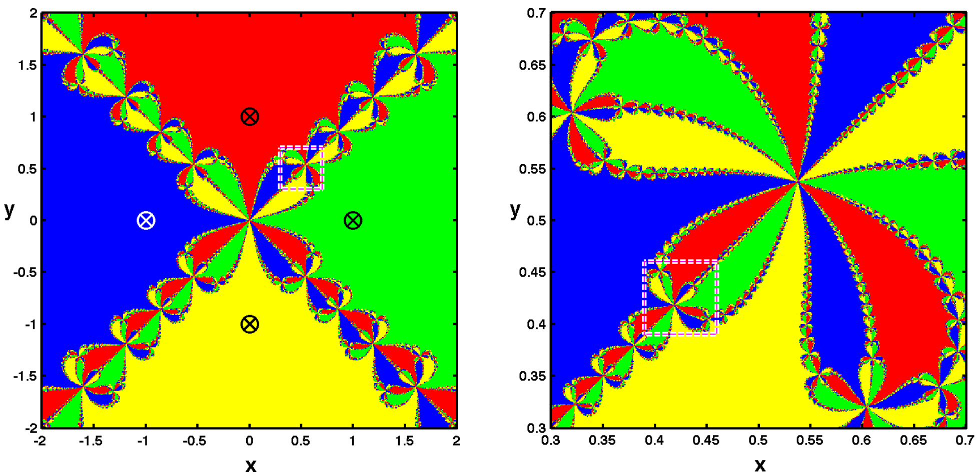
† This replaces zero entries in **Zn** by **del**, because the logical operator **(Zn == 0)** yields zero for all non-zero entries in **Zn**, and one for zero entries.

‡ The result will be $z_{n+1} \approx (1/4)10^{60}$, while for z_n large (2.2) reduces to $z_{n+1} \approx (3/4)\,z_n$. Hence returning to $z_{n+M} = O(1)$ requires, roughly, $(3/4)^M\,10^{60} = O(1)$.

h6. Regions to explore. Do (at least) four figures, exploring the regions: $-2 \leq x,\ y \leq 2$, $-0.3 \leq x,\ y \leq 0.7$, $-0.39 \leq x,\ y \leq 0.46$, and $-0.432 \leq x,\ y \leq 0.436$, where $x = \mathrm{Re}(z_0)$ and $y = \mathrm{Im}(z_0)$.

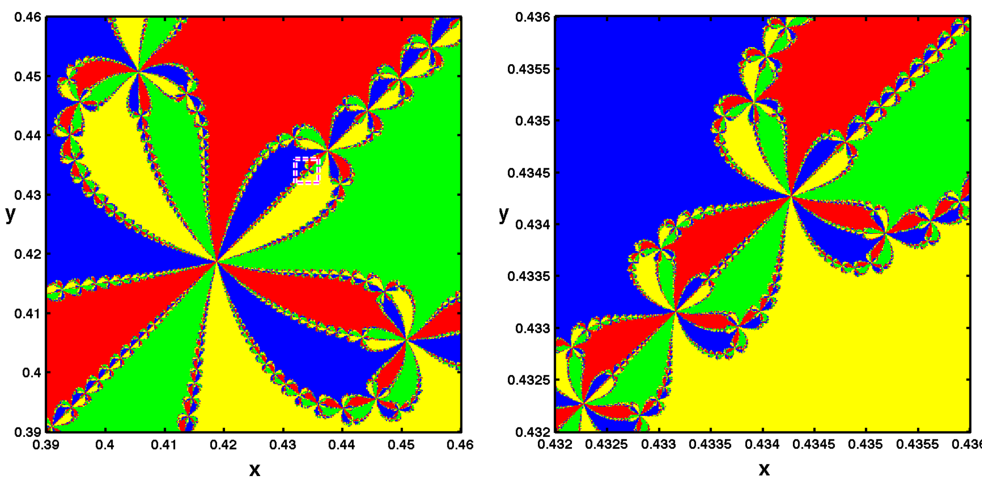
2.2 Answer: Newton's method in the complex plane #02

Figures 2.1–2.2 show the results of our calculations. **In panels (a–c), the square with a white/magenta dashed boundary indicates the region displayed in the next panel.**



Left, (a).
$N = 100$ iterations,
$-2 < x, y < 2$.
Crosses at the roots ζ_j.
Right, (b).
$N = 150$ iterations,
$0.3 < x, y < 0.7$.
Color scheme: (2.4).
Pixels: 500×500.

Figure 2.1: Newton's method in the complex plane #02. Convergence zones for the $z^4 = 1$ Newton's map iterates.



Left, (c).
$N = 200$ iterations,
$0.39 < x, y < 0.46$.
Right, (d).
$N = 250$ iterations,
$0.432 < x, y < 0.436$.
Color scheme: (2.4).
Pixels: 500×500.

Figure 2.2: Newton's method in the complex plane #02. Convergence zones for the $z^4 = 1$ Newton's map iterates.

Note the *fractal nature of the boundary between the basins of attraction for each root:* as we zoom in, the object appears as a smaller (but distorted) copy of itself. Non-trivial self-similarity [2] is the hallmark of a fractal. Sets like this (boundaries between convergence regions of complex analytic iterations) are called **Julia sets**.

> The attracting basins are **Fatou sets**. The sets are named after Gaston Julia and Pierre Fatou, two mathematicians that pioneered the study of complex dynamics — e.g., see: G. Julia, *Mémoire sur l'iteration des fonctions rationnelles,* Journal de Mathématiques Pures et Appliquées, **8:** 47–245, 1918, and P. Fatou, *Sur les substitutions rationnelles*, Comptes Rendus de l'Académie des Sciences de Paris, **164:** 806–808 and vol. 165, pp. 992–995, (1917).

The orbits within the Julia set are chaotic. These orbits are, generally, not periodic (but recurrent), and small differences in z_n grow exponentially with n (sensitive dependence on initial conditions). However, **computing these**

[2] A line in the plane is also self-similar, but it has trivial structure.

orbits is extremely hard, as perturbations out of the Julia set make the resulting orbit convergent.

3 Quasiperiodic functions and Lissajous figures

3.1 Statement: Quasiperiodic functions and Lissajous figures.

Task #1. Using a computer, plot the curve whose parametric equations are $x(t) = \sin(t)$ and $y(t) = \sin(\omega\, t)$, for the following rational and irrational values of the parameter ω.

$$\left.\begin{array}{lll} \textbf{(1)}\ \ \omega = 3 & \textbf{(2)}\ \ \omega = \dfrac{2}{3} & \textbf{(3)}\ \ \omega = \dfrac{5}{3} \\ \textbf{(4)}\ \ \omega = \sqrt{2} & \textbf{(5)}\ \ \omega = \pi & \textbf{(6)}\ \ \omega = \dfrac{1}{2}\left(1+\sqrt{5}\right) \end{array}\right\} \text{Note: (6) is the "golden ratio"} \tag{3.1}$$

Task #2. Do the same for $x(t) = \sin(t)$ and $y(t) = \cos(\omega\, t)$.

The resulting **curves are called Lissajous figures.** They can be displayed on an oscilloscope by using two ac signals of different frequencies as inputs. Note that you will see three qualitatively different types of curves: In the periodic case (i.e.: ω is rational) (i) open curves and (ii) closed curves; and in the true quasi-periodic case (i.e.: ω is irrational) (iii) dense curves in the plotting region.

Task #3, Optional. Can you explain what triggers the difference between (i) and (ii)?

Task #4. Add $\omega = 1$ and $\omega = 1.02$ for the case $x(t) = \sin(t)$ and $y(t) = \cos(\omega\, t)$. Describe what you see.

Task #5 (this task requires no answer). Run the two scripts [QuasiPer2sincos and QuasiPer2sinsin] provided with the problem statement. These two scripts illustrate two important properties of quasi-periodic functions: (1) The "natural" phase space for them is a Torus with as many dimensions as periods they have; and (2) They have "almost periods" $T_n > 0$ over which they repeat with arbitrarily small errors.

Remark 3.2 Obviously, in a numerical calculation ω cannot be "irrational". However, an irrational ω will be approximated by a rational number which is the quotient of two very large integers. The result will be a periodic curve with a very, very, complicated and a very, very long period.

3.2 Answer: Quasiperiodic functions and Lissajous figures

Use the MatLab scripts LissajousSinCos(omega, N) and LissajousSinSin(omega, N) to calculate and plot the Lissajous figures for arbitrary values of ω. In particular, **for ω irrational,** take N large to see the **space filling property of the curve. Tasks #1–2** yield figures 3.1–3.6.

Task #3. In figures 3.1–3.2 we see that $y = \sin(\frac{2}{3}t)$ yields a closed curve, while $y = \sin(3\,t)$ and $y = \sin(\frac{5}{3}t)$ seem to yield open curves. **Closed curves occur because,** at some values $t = t_n$, $\vec{p} = (x(t),\, y(t))$ reverses direction and tracks back the same curve. This is possible because sinusoidals are even relative to extrema (local maximums or minimums). Thus, if both $x(t)$ and $y(t)$ reach an extrema at the same time $t = t_n$, a reversal occurs. The extrema for $x = \sin(t)$ happen for $t = (n + 0.5)\pi$, while the extrema for $x = \sin(\omega\, t)$ happen for $t = (m + 0.5)\pi/\omega$ — n and m integers. Hence, **for a direction reversal to occur, we need $\omega = (2\,m+1)/(2\,n+1)$** — i.e.: ω is the quotient of two odd integers. Similarly, when $y = \cos(\omega\, t)$, a reversal requires $\omega = 2\,m/(2\,n+1)$ — see figures 3.4 and 3.5.

Note that **the curves are algebraic when ω is rational** (i.e.: they can be written in the form $\mathcal{P}(x,\, y) = 0$, where $\mathcal{P}$ is a polynomial). This follows because in this case $y = \sin(\omega\, t)$ (or $y = \cos(\omega\, t)$) is related to $x = \sin(t)$ via trigonometric identities. For example, in figure 3.2, (i) $y = \sin(3\,t)$ yields $y = 3x - 4x^3$. Curve has one y for

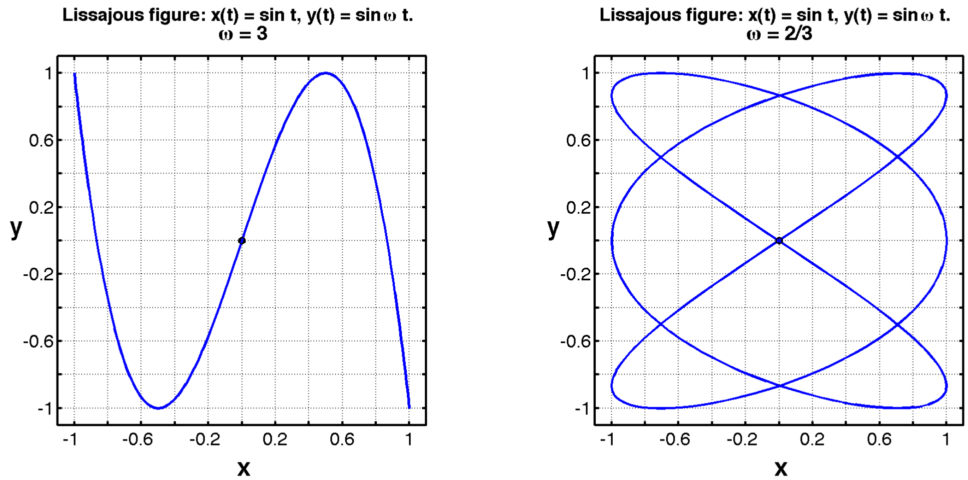


Figure 3.1:
Task #1 (1–2).
$\omega = 3,\ 2/3$.

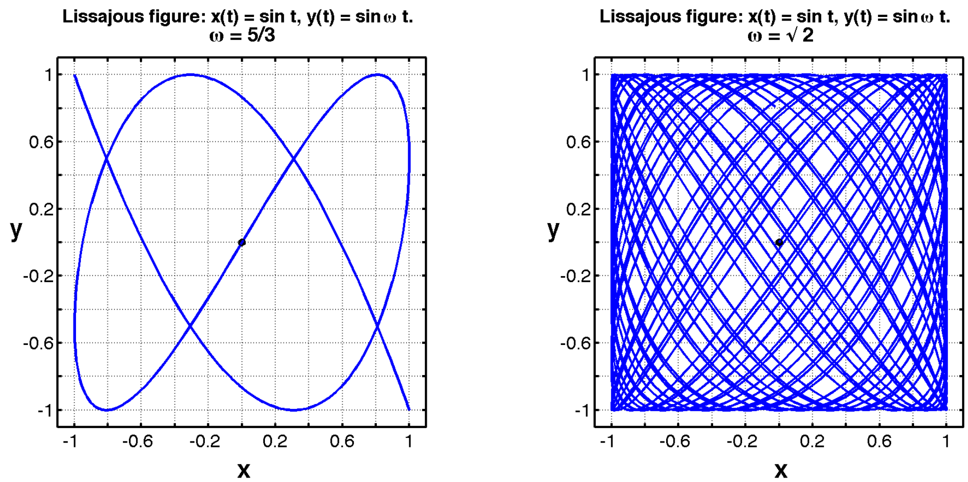


Figure 3.2:
Task #1 (3–4).
$\omega = 5/3,\ \sqrt{2}$.

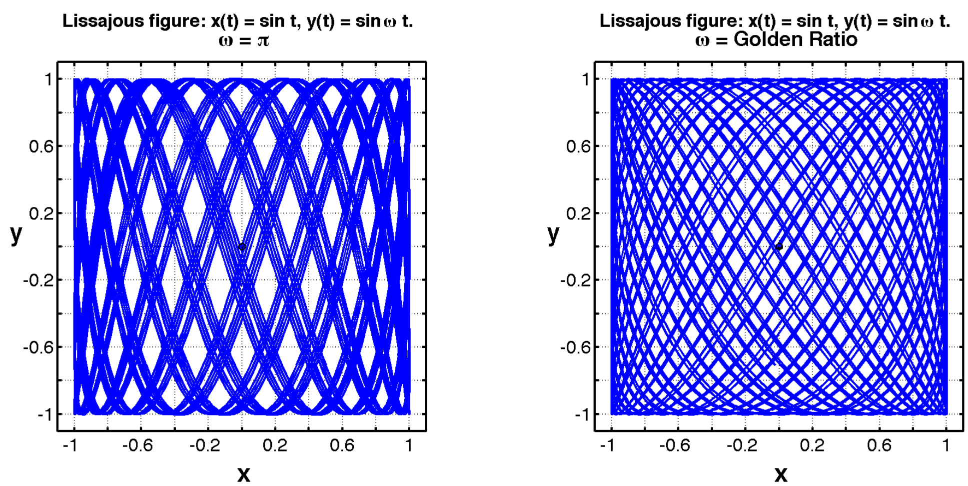


Figure 3.3:
Task #1 (5–6).
$\omega = \pi$, and
$\omega = \frac{1}{2}(1+\sqrt{5})$.

each x, and (generically) three x for each y. (ii) For $y = \sin(\frac{2}{3}\,t)$ yields $4\,y^2(y^2 - \frac{3}{4})^2 = x^2(1-x^2)$. Curve (generically) has six y for each x, and four x for each y.

Task #4. See figure 3.7. For $y = \cos(t)$ the result is, of course, a circle. However, a slight perturbation, to $y = \cos(1.02\,t)$, produces precession — with "almost" circles that do not quite close and induce a slow "rotation" of the curve (in this case counter-clockwise). Of course, because $\mathbf{1.02}$ is rational the curve eventually closes. A small, non-rational, perturbation would induce a never closing precession.

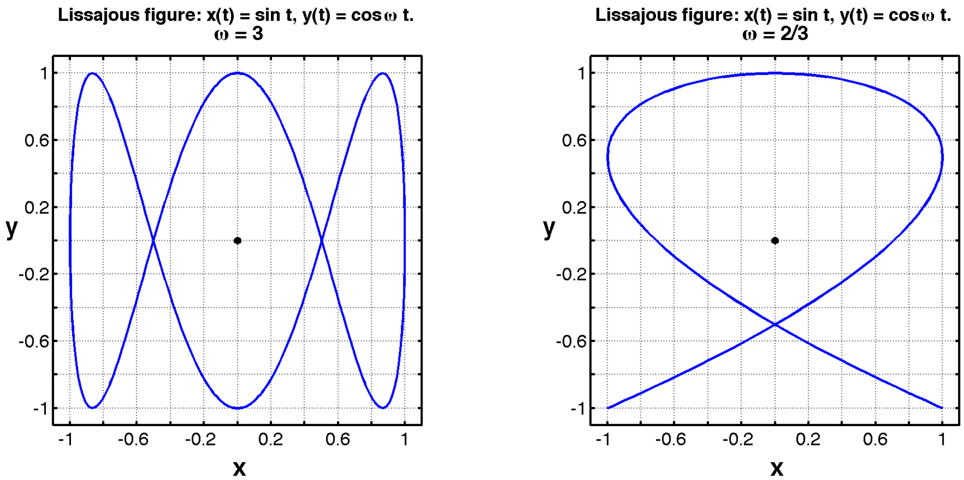


Figure 3.4: **Task #2** (1–2). $\omega = 3,\ 2/3$.

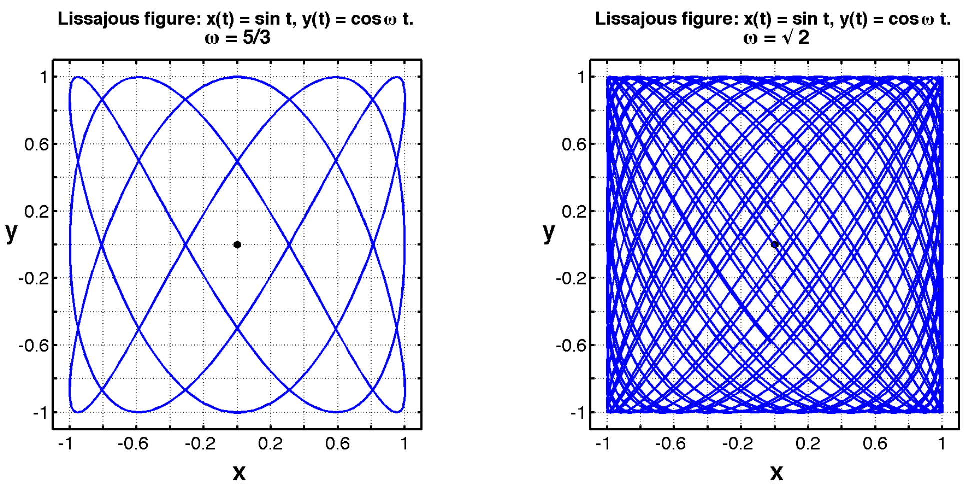


Figure 3.5: **Task #2** (3–4). $\omega = 5/3,\ \sqrt{2}$.

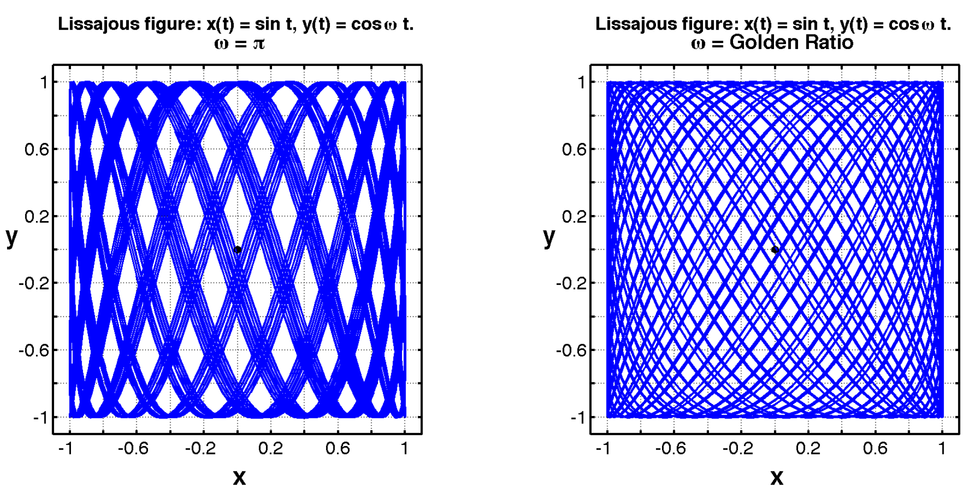


Figure 3.6: **Task #2** (5–6). $\omega = \pi$, and $\omega = \frac{1}{2}(1+\sqrt{5})$.

Finally, note that when ω is irrational the curves are dense in the square $-1 \leq x,\ y \leq 1$. However, this would lead to "solid blue" pictures, where no details of the curve are visible. Thus we have plotted these curves for a not too large range, $0 \leq t \leq t_f$, where t_f is large enough so the fact that the curve will fill the square can be gleaned, but not so large that no details can be seen.

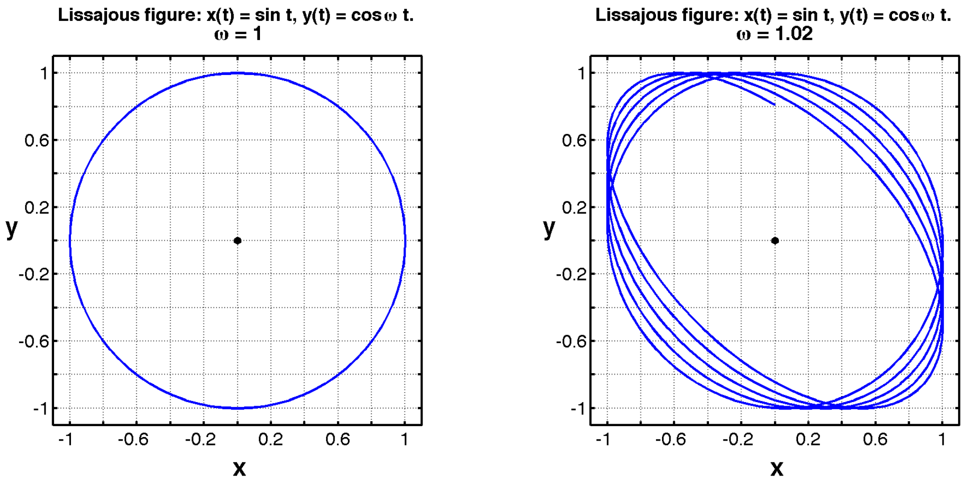


Figure 3.7:
Task #4.
$\boldsymbol{\omega = 1,\ 1.02}$.

4 Sierpinski's carpet

4.1 Statement: Sierpinski's carpet

Consider the process shown in figure 4.1. Divide the closed unit box into nine equal boxes, and delete the open central box. Repeat for each of the eight remaining sub-boxes, and so on. Figure 4.1 shows the first two stages. **Tasks: A.** Sketch the next stage, $\boldsymbol{S_3}$. **B.** Find the similarity dimension of the limiting fractal, known as the **Sierpinski carpet. C.** Show that the Sierpinski carpet has zero area.

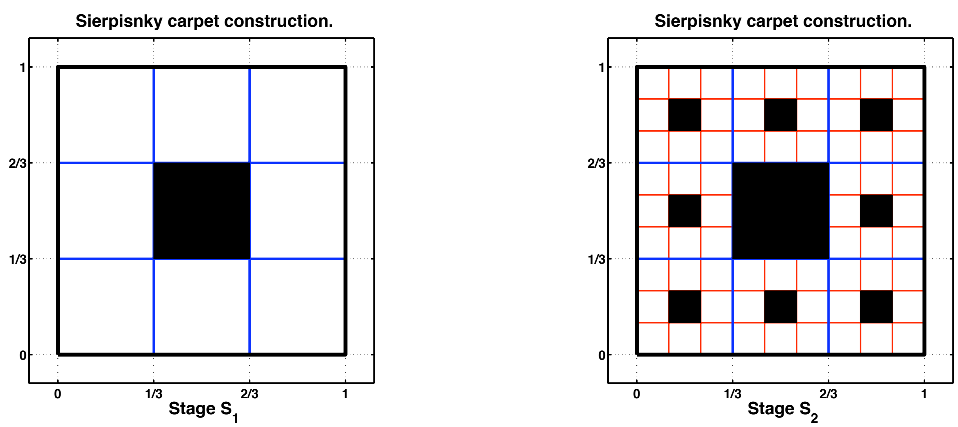


Figure 4.1: **Sierpinski's carpet construction.** The areas shaded in black are the parts of the original square deleted.

4.2 Answer: Sierpinski's carpet

A. Figure 4.2 shows stages S_3 and S_4 in the fractal construction.

B. The carpet is made up by 8 identical copies of itself, each reduced in size by 3. For any $n = 1, 2, 3, \dots$, *the fractal is made by $N = 8^n$ copies, each reduced in size by $r = 3^n$.*
Thus, the **self-similarity dimension d of the fractal** is: $\boldsymbol{d = \frac{\log(N)}{\log(r)} = \frac{\log(8^n)}{\log(3^n)} = \frac{\log(8)}{\log(3)} \approx 1.8928}$.

C. Let A_0 be the area of the box from which the fractal construction is started, and A_n the area of the set S_n. Clearly: $A_{n+1} = \frac{8}{9}\,A_n$, so that $A_n = \left(\frac{8}{9}\right)^n A_0$. Since the fractal is included in **all** the S_n, and $A_n \to 0$ as $n \to 0$, it follows that **the fractal has zero area.**

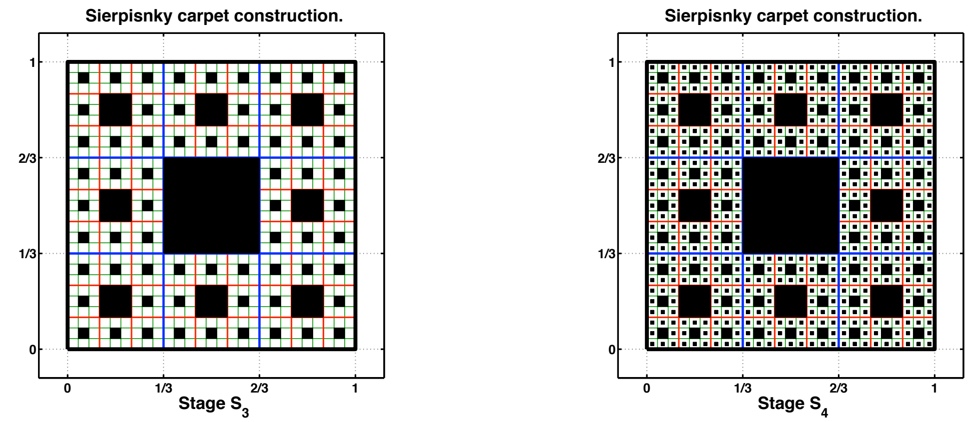


Figure 4.2: Sierpinski's carpet construction. Stages $\boldsymbol{S_3}$ and $\boldsymbol{S_4}$. For $\boldsymbol{S_4}$ we have not drawn the dividing lines at the last level, to avoid cluttering.

Part II
Recovery problems

5 Index for a critical point with zero determinant

5.1 Statement: Index for a critical point with zero determinant

Consider a phase plane system

$$\dot{x} = f(x,\, y) \quad \text{and} \quad \dot{y} = g(x,\, y), \tag{5.1}$$

where f and g are smooth functions of all of its arguments. Assume that:

1. The origin $\mathcal{O}$ is an isolated critical point. That is $f(0,\,0) = g(0,\,0) = 0$, and there are no solutions to $f(x,\,y) = g(x,\,y) = 0$ with $0 < x^2 + y^2 < \epsilon$ — for some ϵ.
2. Let A be the 2×2 matrix corresponding to the linearized system near $\mathcal{O}$, with $\tau = \mathrm{tr}(A)$ and $\Delta = \det(A)$. Suppose that $\boldsymbol{\Delta = 0}$ and $\boldsymbol{\tau > 0}$ — so that one eigenvalue of A vanishes, and the other equals τ.

This is a structurally unstable situation, in particular: the index for $\mathcal{O}$ is not determined at all by the linearized equations. **Construct examples of the above situation where:**

A. $\mathcal{I} = \mathrm{index}(\mathcal{O}) = \quad 1.$

B. $\mathcal{I} = \mathrm{index}(\mathcal{O}) = -1.$

C. $\mathcal{I} = \mathrm{index}(\mathcal{O}) = \quad 0.$

Sketch the phase plane portraits for the systems that you construct.

Hints. Consider the linear system $\dot{Y} = A\,Y$, and then add a nonlinear correction which:

For part **A**. Makes $\mathcal{O}$ into a (nonlinear) node.

For part **B**. Makes $\mathcal{O}$ into a (nonlinear) saddle.

For part **C**. Makes $\mathcal{O}$ into a (nonlinear) saddle on one side, and a (nonlinear) node on the other.

Note that you can do **A**-**B** by writing a linear system $\dot{Y} = \tilde{A}\,Y$ where $\tilde{A}$ gives you either a node or a saddle, and then replacing the entries that vanish in A by appropriate nonlinear terms that vanish at the origin. **C** is a little more involved, so first do **A**-**B**.

5.2 Answer: Index for a critical point with zero determinant

We take A of the form
$$A = \begin{pmatrix} 0 & 0 \\ 0 & \tau \end{pmatrix}, \tag{5.2}$$
and then follow the strategy in the hints:

A. Let
$$\dot{x} = x^3 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{5.3}$$
This system has a (nonlinear) node at the origin. It, effectively, behaves as a "linear" system with two positive, and non-equal, eigenvalues: $\lambda = \tau$ and $\lambda = x^2$. Hence $\boldsymbol{\mathcal{I} = 1}$. The phase plane portrait for this system can be found on the left panel of figure 5.1.

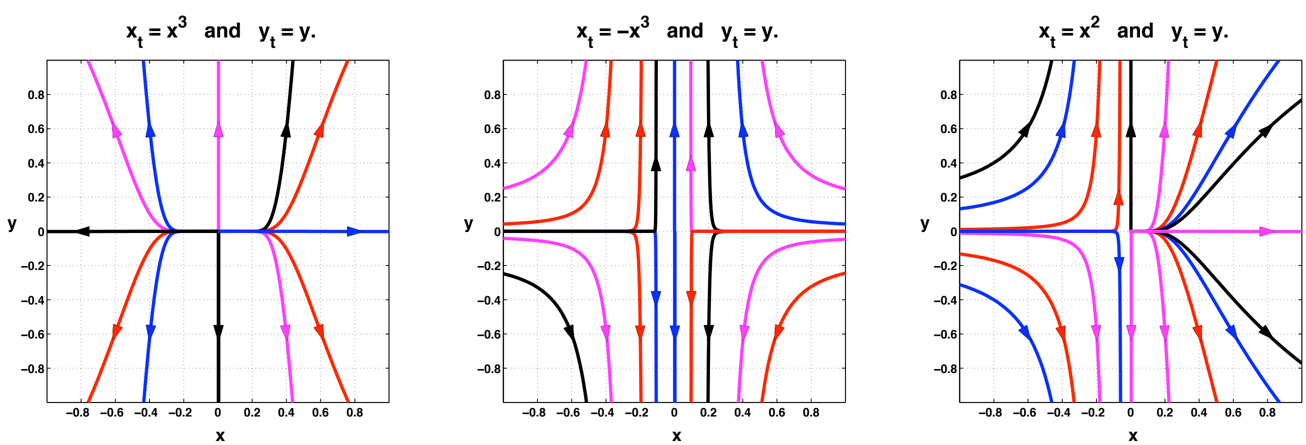


Figure 5.1: Phase plane portraits for the systems in (5.3 – 5.5) — ordered left to right.

B. Take
$$\dot{x} = -x^3 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{5.4}$$
This system has a (nonlinear) saddle at the origin. It, effectively, behaves as a "linear" system with eigenvalues: $\lambda = \tau > 0$ and $\lambda = -x^2 < 0$. Hence $\boldsymbol{\mathcal{I} = -1}$. The phase plane portrait for this system can be found on the middle panel of figure 5.1.

C. Take
$$\dot{x} = x^2 \quad \text{and} \quad \dot{y} = \tau\, y. \tag{5.5}$$
This system, effectively, behaves as a "linear" system with eigenvalues: $\lambda = \tau > 0$ and $\lambda = x$. Hence, for $x > 0$ node-like behavior occurs, while for $x < 0$ saddle-like behavior occurs. It follows that $\boldsymbol{\mathcal{I} = 0}$. The phase plane portrait for this system can be found on the right panel of figure 5.1.

6 Limit cycle bifurcation with a computer #01

6.1 Statement: Limit cycle bifurcation with a computer #01

The system
$$\dot{x} = y/\delta \quad \text{and} \quad \dot{y} = (2\,\mu\,y - (1+\mu^2)\,x + x^2\,(\mu\,x - y))/\delta, \tag{6.1}$$
where $\delta = 1 + x^2$, undergoes a supercritical Hopf bifurcation at $\mu = 0$. Beyond that, as μ increases, while the critical points at $(x, y) = (\pm x_0,\, 0)$ ($x_0 = \sqrt{\mu + 1/\mu}$) move towards the origin. For $\mu = \mu_c$, $0.3 < \mu_c < 0.4$, **another bifurcation occurs.** Using a computer:

(1) Describe what happens at $\mu = \mu_c$.

(2) Find the nature of all the critical points of the system.

(3) Ascertain the value of μ_c with two significant digits.

Illustrate your conclusions with three phase plane portraits in the region $-3 < x,\ y < 3$, one for $\mu < \mu_c$, one for $\mu > \mu_c$, and one for $\mu \approx \mu_c$.

Remark #1. Note that the system has the symmetry $(x,\ y) \to (-x,\ -y)$.

Suggestions for computation. Since the problem involves investigating a situation as a parameter varies, with an equation with many terms, PHPLdemoB is not the best tool (impractical). Instead, I suggest that you use **PHPLdemoB2**, which will allow you to do μ "sweeps" efficiently. Once you know well what happens, to produce "neat" final plots, you could use **PHPLplot** or **PHPLplot_v2**, which are less user friendly than PHPLdemoB2, but allow precise control of the orbits.

6.2 Answer: Limit cycle bifurcation with a computer #01.

The critical points of are the **origin (unstable spiral for $\mu > 0$) and $(x, y) = (\pm x_0,\ 0)$** ($x_0 = \sqrt{\mu + 1/\mu}$), both of which are **saddles**. This can be seen in all the panels in figure 6.1. For $0 < \mu < \mu_c$, left panel in figure 6.1, the

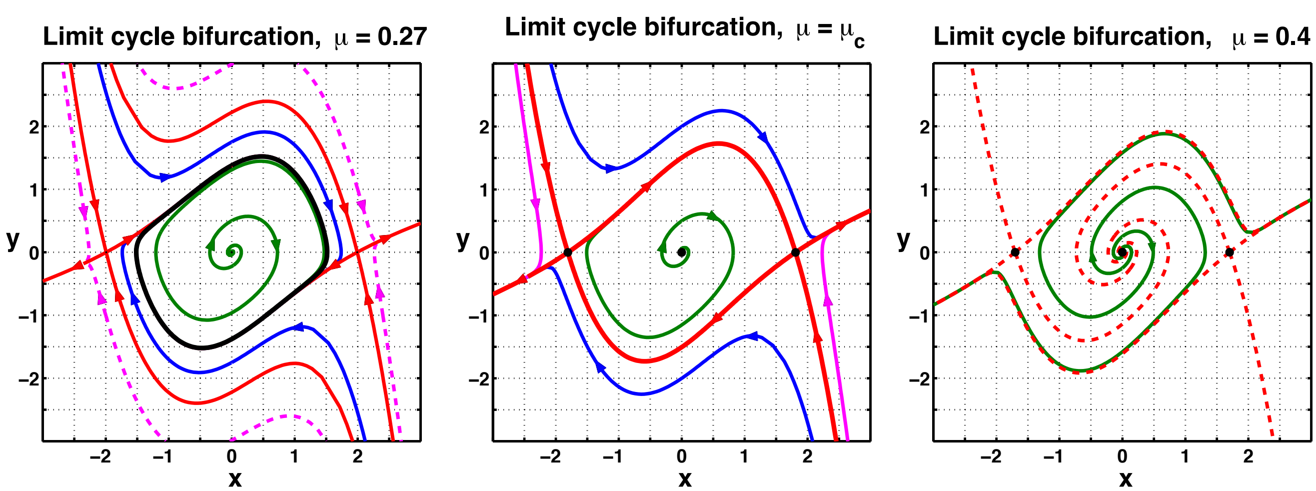


Figure 6.1: Limit cycle bifurcation with a computer #01, for the system in (6.1). **Left panel:** $\mu = 0.27 < \mu_c$ **Middle panel:** $\mu = \mu_c = 0.34106\ldots$ **Right panel:** $\mu = 0.40 > \mu_c$.

limit cycle encloses the origin and it is located in between the two saddles — see item **(1)** below. As μ increases, the limit cycle gets larger, while the saddles move towards the origin (in this range x_0 is a decreasing function of μ). When μ reaches μ_c, where $0.34106 < \mu_c < 0.34107$, the two saddles merge with the limit cycle (simultaneously, because of the symmetry in remark #1), as shown in the middle panel of figure 6.1 — see item **(2)** below. Beyond μ_c, $\mu > \mu_c$, there is no limit cycle, as shown in the right panel of figure 6.1 — see item **(3)** below.

(1) Figure 6.1, left panel, $\mu < \mu_c$. Here the stable/unstable trajectories of the saddle (in red) divide the plane into seven regions. The trajectories inside the limit cycle (in green) approach the limit cycle (in black) as $t \to \infty$, and the origin as $t \to -\infty$. The trajectories in the regions above and below the limit cycle (in blue) approach the limit cycle as $t \to \infty$, and diverge to infinity as $t \to -\infty$. The trajectories in the four remaining regions (in magenta) diverge to infinity as both $t \to \pm\infty$.

(2) Figure 6.1, middle panel, $\mu = \mu_c$. The **limit cycle** is destroyed, and **becomes a cycle graph,** made by two heteroclinic orbits connecting the saddles. Thus we have an **heteroclinic bifurcation of the limit cycle,** very **similar to an homoclinic bifurcation** (with one saddle merging with the cycle), **possible only because of the symmetry in remark #1** — any break of symmetry would destroy the simultaneous nature of the "collisions" of the saddles with the limit cycle. The trajectories inside the cycle graph approach the cycle graph as $t \to \infty$, and the origin as $t \to -\infty$. The other trajectories (except for the saddle's stable/unstable manifolds) all diverge to infinity as $t \to \pm\infty$.

(3) Figure 6.1, right panel, $\mu > \mu_c$. There is **no limit cycle** and all the trajectories (except for the saddle's stable manifolds) escape to infinity as $t \to \infty$. As $t \to -\infty$ some trajectories approach the origin, while the others diverge to infinity (again, excluding the saddle's unstable manifolds).

7 Lorenz equations: linear stability for two CP

7.1 Statement: Lorenz equations: linear stability for two CP

Consider the **Lorenz equations,** $\dot{x} = \sigma\,(y - x)$, $\dot{y} = r\,x - y - x\,z$, $\dot{z} = x\,y - b\,z$, where $\sigma,\ b > 0$ and $r > 1$.

1. Show that the fixed points for the Lorenz equations are: $C^{\pm} = (\pm\sqrt{b\,(r-1)},\ \pm\sqrt{b\,(r-1)},\ r-1)$ and the origin. Note that if $r \leq 1$ the only fixed point is the origin.
2. Show that the characteristic polynomial for the eigenvalues of the Jacobian matrix at $C^{\pm}$ is
$$\lambda^3 + (\sigma + b + 1)\,\lambda^2 + (r + \sigma)\,b\,\lambda + 2\,b\,\sigma\,(r-1) = 0.$$
3. By seeking solutions of the form $\lambda = i\,\omega$, with ω real and non-zero, show that there is a pair of purely imaginary eigenvalues when
$$r = r_H = \sigma\left(\frac{\sigma + b + 3}{\sigma - b - 1}\right), \quad \text{provided that } \sigma > b + 1.$$
The value r_H is where a subcritical Hopf bifurcation occurs. *Explain why we need to assume $\sigma > b + 1$.*

7.2 Answer: Lorenz equations: linear stability for two CP

1. From the first equation, $x = y$ at a fixed point. The other two equations then reduce to (i) $x\,(r - 1 - z) = 0$ and (ii) $b\,z = x^2$. From (i), either: (a) $x = 0$, in which case $x = y = z = 0$; or (b) $z = r - 1$, so that $x = \pm\sqrt{b\,(r-1)}$, which leads to $C^{\pm}$. Obviously, $C^{\pm}$ require $r \geq 1$, so that x and y are real.

   Note now that the Lorenz equations are symmetric under $(x,\, y,\, z) \to (-x,\, -y,\, z)$. **Thus, for the rest of the problem it is sufficient to consider C^+ only.**

2. The Jacobian matrix, at any point $P = (x,\, y,\, z)$ is:
$$\mathcal{A}(P) = \begin{pmatrix} -\sigma & \sigma & 0 \\ r - z & -1 & -x \\ y & x & -b \end{pmatrix}. \quad \text{Let } \mathcal{A}^{\pm} = \mathcal{A}(C^{\pm}).$$

Then the characteristic equation is:

$$0 = \det(\mathcal{A}^+ - \lambda\mathcal{I}) = \det \begin{pmatrix} -\sigma-\lambda & \sigma & 0 \\ 1 & -1-\lambda & -\sqrt{b\,(r-1)} \\ \sqrt{b\,(r-1)} & \sqrt{b\,(r-1)} & -b-\lambda \end{pmatrix}$$

$$= (-\sigma-\lambda)(-1-\lambda)(-b-\lambda) - \sigma\, b\,(r-1) + b\,(r-1)\,(-\sigma-\lambda) - (-\sigma-\lambda)\,\sigma$$

$$= -\lambda^3 - (\sigma+b+1)\,\lambda^2 - (r+\sigma)\,b\,\lambda - 2\,b\,\sigma\,(r-1).$$

3. Substitute $\lambda = i\,\omega$ into the characteristic equation. Then the real and imaginary parts yield $\omega^3 = \omega\, b\,(r+\sigma)$ and $(\sigma+b+1)\,\omega^2 = 2\,b\,\sigma\,(r-1)$. Divide the first equation by ω (since $\omega = 0$ is **not** a solution of the second equation), substitute the first equation into the second, and solve for r to obtain
$$\omega^2 = b\,(r+\sigma) \quad \text{and} \quad r = r_H = \sigma\frac{\sigma+b+3}{\sigma-b-1}.$$
This only makes sense if $\sigma > b+1$, otherwise r_H is either negative or infinite. Once $\sigma > b+1$, it is easy to check that $r_H > 1$.

THE END.