Answers to P-Set # 06, (18.353/12.006/2.050)j MIT (Fall 2024)

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1.1 Statement: Degenerate bifurcation - fails to be Hopf

Consider the damped/driven Duffing oscillator
$$\ddot{x} + \mu \dot{x} + x - x^3 = 0, \qquad (1.1)$$
where μ is a constant.

a) Show that the origin changes from a stable to an unstable spiral as μ decreases through zero.

b) Plot the phase portraits for $\mu > 0$, $\mu = 0$ and $\mu < 0$, and show that the bifurcation at $\mu = 0$ is a degenerate version of the Hopf bifurcation. In fact, **show that no periodic orbits are possible for $\mu \neq 0$.**
The nonlinearity is neither stabilizing, nor destabilizing. A "degenerate", very special, case. Hint: Energy/Liapunov.

1.2 Answer: Degenerate bifurcation - fails to be Hopf

a) The linearized equation near the critical point at the origin is $\ddot{x} + \mu\dot{x} + x = 0.$ (1.2)
This is a damped/driven harmonic oscillator for $\mu > 0/\mu < 0$,
corresponding to a stable/unstable critical point: spiral for $0 < \mu^2 < 4$, and node for $\mu^2 > 4$.

b) The switch from stable to unstable spiral at the origin suggests that a Hopf bifurcation might occur at $\mu = 0$. However, the energy

$$E = \frac{1}{2}(\dot{x})^2 + \frac{1}{2}x^2 - \frac{1}{4}x^4, \quad (1.3)$$

satisfies

$$\dot{E} = -\mu\,(\dot{x})^2. \quad (1.4)$$

In particular, E is a **conserved quantity** for $\mu = 0$. This shows that the nonlinear terms in equation (1.1) are conservative, neither stabilizing nor destabilizing. This is a necessary condition for a Hopf bifurcation, since the limit cycle that arises there is the product of the balance between the linear and the nonlinear terms. We conclude that: **no Hopf bifurcation happens at $\mu = 0$.**

Another indication that a Hopf bifurcation cannot occur for $\mu = 0$ is: If there was one, there would be a limit cycle for only one of either $\mu > 0$ or $\mu < 0$. But equation (1.1) is invariant under the transformation $t \to -t$ and $\mu \to -\mu$, which is not compatible with the existence of a limit cycle on only one side.

We confirm the conclusion in the paragraphs above by proving that **no periodic orbits are possible for $\mu \neq 0$.** For any periodic orbit:

$$0 = \oint \dot{E}\,dt = -\mu \oint (\dot{x})^2\,dt. \quad (1.5)$$

Thus $\dot{x} \equiv 0$, which means that the orbit is actually the critical point.

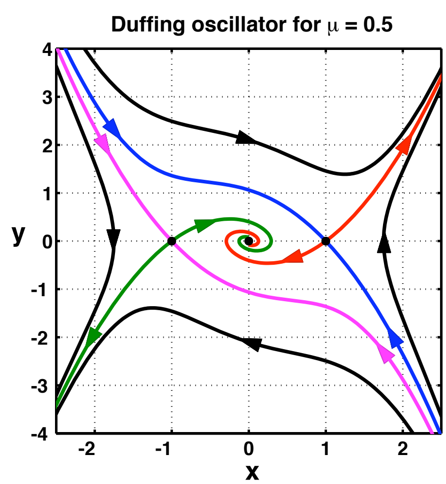


Figure 1.1: Phase portrait for (1.1), with $\mu = 0.5$. The saddle's stable/unstable manifolds are shown. Here $y = \dot{x}$.

To study the **phase portrait for equation (1.1),** we note that we **need to consider only the case $\mu > 0$**, because the equation is invariant under $t \to -t$ and $\mu \to -\mu$. Hence the phase portrait for $\mu = -\gamma < 0$ is the same as the one for $\mu = \gamma$ with the arrows reversed. We also note that the phase portrait (for any μ) must be invariant under $(x,\, \dot{x}) \to -(x,\, \dot{x})$.

For $0 < \mu \ll 1$ the phase portrait (near the critical point) for equation (1.1) can be obtained by noticing that E is a slowly decreasing function of time. Thus the orbits are "almost" level lines for E. In particular, the behavior of the stable/unstable manifolds for the saddles at $P_1 = (x,\, \dot{x}) = (-1,\, 0)$ and $P_2 = (x,\, \dot{x}) = (1,\, 0)$ follows.[1] For example, one of the unstable manifolds for each saddle will have to approach the critical point at the origin, while the other must diverge to infinity. Figure 1.1 shows the case $\mu = 0.5$.

[1] Note that, once we have these, the behavior of the other orbits is pretty much determined.

2 Dog and duck in a pond

2.1 Statement: Dog and duck in a pond

A dog at the center of a circular pond sees a duck swimming along the edge. The dog chases the duck by always swimming straight towards it. In other words, the dog's velocity vector always lies along the line connecting it to the duck. Meanwhile, the duck takes evasive action by swimming around the circumference as fast as it can, always moving counterclockwise.

Task #1. Assume that the pond has unit radius, and that both animals swim at the same constant speed, and **derive a pair of differential equations for the dog path.** Use a system of coordinates with the origin at the pond's center — see figure 2.1, and write equations for $\frac{dR}{d\theta}$ and $\frac{d\phi}{d\theta}$.
Notation: (i) R is the *distance between the dog and the duck*, **(ii)** θ is the polar angle for the duck's position, and **(iii)** ϕ is the angle (measured counter-clockwise) between the lines connecting the duck's position to the center of the pond, and the line connecting the duck's position to the dog's position. See figure 2.1. **Further: (iv)** let $v > 0$ be

Dog-duck coordinate system.

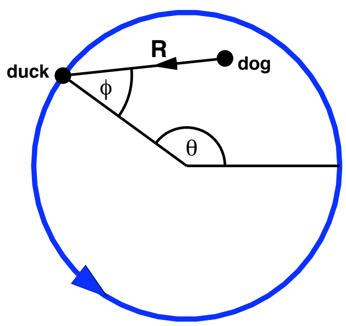


Figure 2.1: Dog and Duck in a pond
The Dog-Duck coordinate system; with the variables R, ϕ, and θ.

the *constant linear speed of the dog*, **(v)** let $s > 0$ be the *constant angular speed of the duck* — hence $\theta = s\,t + \theta_0$, for some constant θ_0, and **(vi)** let $k = v/s$ be the *ratio of the dog's speed to the duck's speed.*
The independent variable in the equations will be the angle θ, not time. However, because $\theta = s\,t + \theta_0$, θ and t are equivalent.

Hint #1. Because the motion occurs in the plane, using complex numbers to represent vectors is useful. Thus, let $D = D(t) =$ *position of the dog* at time t, and let $d = d(t) = e^{i\,\theta} =$ *position of the duck* at time t. The *vector connecting the dog to the duck* is then $d - D = R\,e^{i\,\psi}$, where ψ can be easily written in terms of θ and ϕ.

Hint #2. Notice that, along the edge of the pond, the swimming direction for the dog would always be towards the inside of the pond, except when/if the dog catches the duck (where ϕ is no longer defined, and the dog-duck coordinate system is singular). Thus *the dog stays inside the pond till/if it captures the duck, with* $-\pi/2 < \phi < \pi/2$. This information will be useful when answering the questions below.

Task #2. Answer the question: what value does ϕ approach when/if the dog captures the duck? That is, as $R \downarrow 0$?
Hint #3. Look at the equation for $\frac{d\phi}{d\theta}$.

Task #3. Show that there is a critical $k_c > 0$ such that: (i) If $k > k_c$ the *dog catches the duck in a finite time.* (ii) If $k < k_c$, the *dog never catches the duck.* Note that k_c is a simple number, the same one an intuitive argument suggests — the problem, however, requires more than an intuitive argument.

Hint #4. The case $k > k_c$ is easy to analyze; just look at the equation for $\frac{\mathbf{d}\boldsymbol{R}}{\mathbf{d}\boldsymbol{\theta}}$.
To show that there can be no capture for $k < k_c$, use the result in *Hint #2.*

Task #4 (optional beyond finding and clasifying the critical point(s)). Give a complete description of the phase plane (ϕ, R) for the case $0 < k < k_c$. In particular, show that there is a critical point that is a global attractor, corresponding to a final situation where the dog swims on a circle of radius k, at a constant distance from the duck. A complete answer here is challenging; you will have to: (i) use Dulac's criterion to rule out limit cycles, and (ii) construct trapping regions and use the Poincaré-Bendixon theorem to show that all the orbits end up at the critical point.

Task #5 (optional challenge question). What happens for $\boldsymbol{k = k_c}$?

2.2 Answer: Dog and duck in a pond

Note. This problem has a long history, dating back to the mid-1800s at least. It is more difficult than similar **pursuit problems**; for example: there is no known solution for the dog's path in terms of elementary functions. See pp. 113–125 in Davis, H. T., *Introduction to Nonlinear Differential and Integral Equations* (Dover, New York, 1962) for a nice analysis and a guide to the literature. The answer below shows that, even though the problem is beyond the reach of exact solution approaches, it is easily treatable with the tools developed in this course. ♣

Following *hint #1*, we use complex numbers to represent the vectors. In particular, note that $\boldsymbol{\psi = \theta + \phi}$ (move the vector $d - D$ so that its base coincides with the duck's position).

#1 Governing equations. Because the dog's velocity vector always lies along the line connecting it to the duck, we have:

$$\frac{d\vec{D}}{dt} = v\,\frac{\vec{d} - \vec{D}}{R} = v e^{i\psi}. \tag{2.1}$$

Substituting $D = d - Re^{i\psi}$ and $\psi = \theta + \phi$, multiplying by $e^{-i\psi}$, and separating the real from the imaginary parts yields:

$$\frac{dR}{dt} = s\sin(\phi) - v \qquad \text{and} \qquad \frac{d\phi}{dt} = \frac{s}{R}\cos(\phi) - s. \tag{2.2}$$

Since $d\theta = s\,dt$, we obtain:

$$\frac{dR}{d\theta} = \sin(\phi) - k \qquad \text{and} \qquad \frac{d\phi}{d\theta} = \frac{1}{R}\cos(\phi) - 1. \tag{2.3}$$

These equations apply for $\boldsymbol{R > 0}$ (physical range), and *are valid even for non-constant animal speeds.*

#2 Capture angle. *Assume that the dog captures the duck.* Then, as $R \downarrow 0$, the second equation in (2.3) shows that the rate of change for ϕ becomes very large away from the zeros of the cosine. Since $-\pi/2 < \phi < \pi/2$, it should then be clear that $\boldsymbol{\phi \uparrow \pi/2}$. Hence *the dog always captures the dog from behind.*

#3 Critical ratio $\boldsymbol{k_c}$. Below we show that $\boldsymbol{k_c = 1}$, by analyzing the cases $k > 1$ and $k < 1$. Notice that the answer $k_c = 1$ *would be obvious if the animals were swimming to optimize their chances in an unconstrained environment.* Here, however, the duck cannot swim away from the dog because of the pond's edge (he could fly, though) and the dog is pursuing a foolish strategy: he could anticipate the duck's position when choosing a swimming direction. Hence, it is not clear that $k_c = 1$ should apply in this situation also (but it does).

- **3a Case $\boldsymbol{k > 1}$.** Then $\frac{\mathrm{d}R}{\mathrm{d}\theta} \leq 1 - k < 0$. Hence the *dog catches up to the duck (i.e.: $R = 0$) in a finite time.*
- **3b Case $\boldsymbol{0 < k < 1}$.** From the result in item **#2**, it should be obvious that there can be no capture in this case. If there were, then as R vanishes, ϕ would approach $\pi/2$, hence $\frac{\mathrm{d}R}{\mathrm{d}\theta}$ would approach $1 - k > 0$, which contradicts the vanishing of R.

#4 Phase portrait for $\boldsymbol{0 < k < 1}$ (figure 2.2 right). From the equations, it should be clear that in this case there is a critical point at $\boldsymbol{(\phi_1,\, R_1) = (\arcsin(k),\, \sqrt{1 - k^2})}$, and that this critical point is a **stable spiral.** Further, **there are no other critical points for $\boldsymbol{-\pi/2 < \phi < \pi/2}$ and $\boldsymbol{0 < R}$.**[†] Below we show that,

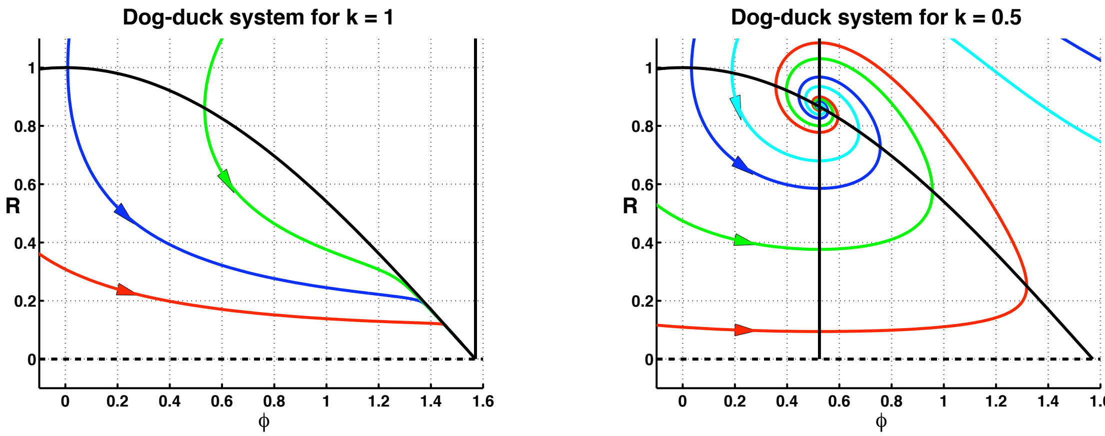


Figure 2.2: Phase plane for the dog and duck system. Nullclines = black lines. $R = 0$ singular line = dashed black line. **Left:** $k = 1.0$, nullclines $R = \cos(\phi)$ and $\phi = \frac{\pi}{2}$. **Right:** $k = 0.5$, nullclines $R = \cos(\phi)$ and $\phi = \arcsin(k)$.

as $\theta \longrightarrow \infty$, the solutions approach this point. (2.4)

† There is also a critical point, an unstable spiral, at $(\phi_2,\, R_2) = (\pi - \arcsin(k),\, -\sqrt{1-k^2})$. But this is outside the physical range.

Now note that

4a There is a trapping region, given by hint #2, and the fact that there is no capture in this case. This region includes the initial point $(\phi,\, R) = (0,\, 1)$, the critical point $(\phi_1,\, R_1)$, and no other critical point.

4b The equations in (2.3) do not have limit cycles. This follows from Dulac's criteria, since

$$\mathrm{div}\left(R\left(\frac{\mathrm{d}\phi}{\mathrm{d}\theta},\, \frac{\mathrm{d}R}{\mathrm{d}\theta}\right)\right) = -k < 0.$$

Then the Poincaré Bendixon theorem gives the result in (2.4).

Notice that **in this case the asymptotic path that the dog follows is given by:**

$$D = e^{i\theta} - R_1 e^{i\psi_1} = \left(1 - R_1 e^{i\phi_1}\right) e^{i\theta} = \left(k^2 - ik\sqrt{1-k^2}\right) e^{i\theta} = k e^{i(\theta - \varphi)},$$

where (obviously) $\psi_1 = \theta + \phi_1$ and $k - i\sqrt{1-k^2} = e^{-i\varphi}$. Thus, **the dog ends up swimming in a circle of radius k, with angle φ, behind the duck.**

#5 Case $k = 1$. *The dog catches up to the duck, but only as $t \to \infty$* (i.e.: never). We show this next:

Analytical proof. Since $\frac{\mathrm{d}R}{\mathrm{d}\theta} \le 0$, R is monotone non-increasing. Thus, if $R > 0$ for all t (the dog does not reach the duck in a finite time),

$$R \to R_\infty \quad \text{as} \quad \theta \to \infty, \tag{2.5}$$

where[2] $0 \le R_\infty \le 1$. Now, if $R_\infty > 0$, the equation on the right in (2.3) shows that $\phi \to \phi_\infty$, where $\cos(\phi_\infty) = R_\infty$. But then, from the equation on the left in (2.3),

$$\frac{\mathrm{d}R}{\mathrm{d}\theta} \to \sin(\phi_\infty) - 1 = \pm\sqrt{1 - R_\infty^2} - 1 < 0,$$

which contradicts (2.5). Thus, it must be $R_\infty = 0$.

We conclude that the dog reaches the duck in either a finite or an infinite time. Unfortunately it is rather hard (with this analysis) to figure out which of the two actually happens.

Geometrical proof. *A better understanding of what happens can be obtained by looking at the* (ϕ, R) *phase plane* (see figure 2.2 left). Note that the region Ω defined by: $|\phi| < 0.5\,\pi$ and $0 < R < \cos(\phi)$ is a **trapping region.** This follows easily by checking the direction of the flow on the nullcline $R = \cos(\phi)$ and noticing that:

[2] Since the dog starts at the center of the pond, $R = 1$ initially. Thus $R \le 1$ for all times.

when R gets small in Ω, $\frac{d\phi}{d\theta} \to \infty$. Furthermore: *the solution of interest is inside* Ω *for all* $t > 0$, since it starts on $(\phi,\, R) = (0,\, 1)$.

Since inside Ω both: $\frac{dR}{d\theta} < 0$ and $\frac{d\phi}{d\theta} > 0$, it follows that all the solutions must eventually approach the point $(\phi,\, R) = (\pi/2,\, 0)$. However, as pointed out above, when R gets small in Ω, $\frac{d\phi}{d\theta} \to \infty$. Thus, as R gets smaller, the solution gets closer and closer to the nullcline $R = \cos(\phi)$, where $\frac{d\phi}{d\theta} = 0$ (this is the same type of argument used for the van der Pol equation limit cycle in the relaxation oscillations regime). It follows that the final approach to
the point $(\phi,\, R) = (\frac{\pi}{2},\, 0)$ satisfies: $\phi \approx \arccos(R)$ and $\frac{dR}{d\theta} \approx \sqrt{1 - R^2} - 1 \approx -\frac{1}{2}R^2$.
Hence $R \approx 2\,\theta^{-1}$ as $\theta \to \infty$. Thus:
the dog catches the duck only as $t \to \infty$.

3 Hopf bifurcation using a computer #02

3.1 Statement: Hopf bifurcation using a computer #02

For the following system

$$\frac{dx}{dt} = \mu\, x + y - x^3 \qquad \text{and} \qquad \frac{dy}{dt} = -x + \mu\, y + 2\, y^3, \tag{3.1}$$

a Hopf bifurcation occurs at the origin when $\mu = 0$. Using a computer, **plot the phase portrait and determine whether the bifurcation is subcritical or supercritical. For small values of μ, verify that the limit cycle is nearly circular. Then measure the period and radius of the limit cycle, and show that the radius R scales with μ as predicted by theory.**

Hints: (a) The nonlinear terms determine the nature of the bifurcation: supercritical if they stabilize, subcritical if they de-stabilize. Hence *check the behavior of the orbits near the critical point for a very small value of* μ *to see if the nonlinear terms stabilize or de-stabilize.* **(b)** In the *subcritical case* the limit cycle is unstable, very hard to compute forward in time. However, if you *run the system backwards in time the solutions converge to the limit cycle.* Note that the PHPL scripts include computations both backards and forwards in time.

3.2 Answer: Hopf bifurcation using a computer #02

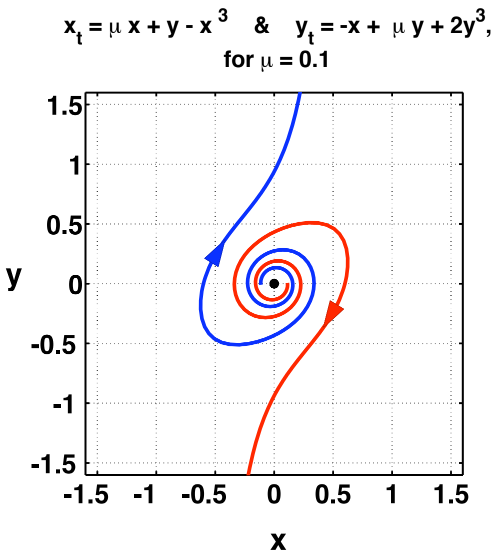
See figure 3.1 for the phase portrait with $\mu = 0.1$. It indicates a destabilizing nonlinearity, thus a **subcritical (hard) Hopf bifurcation.**

Figure 3.2 shows a picture of the phase portrait for $\mu = -0.1$ on the left and the (unstable) limit cycles for various negative values of $0 < -\mu \ll 1$ on the right. An unstable limit cycle appears around the critical point for $0 < -\mu \ll 1$. Thus **a subcritical (hard) Hopf bifurcation** occurs.

For $0 < -\mu \ll 1$ it is easy to see (in figure 3.2) that the limit cycles are nearly circular. The table on the right shows a listing of various parameters for these cycles. Clearly the theoretical prediction (e.g.: $R \sim$ constant $\sqrt{|\mu|}$, and period $\sim$ linear period) is satisfied.

Limit Cycle Parameters, $0 < -\mu \ll 1$.

μ	R =radius.	$\frac{R}{\sqrt{\lvert\mu\rvert}}$	$\frac{\text{Period}}{\pi}$.
-0.016	0.2065	1.6325	2.0032
-0.008	0.1461	1.6334	2.0008
-0.004	0.1033	1.6333	2.0002



Hopf bifurcation using a computer #02.
The system in (3.1) for $\mu = 0.1 > 0$.
All the orbits spiral away from the origin, even those $O(1)$ away. Since μ is fairly small, this is a good hint that the nonlinearity is de-stabilizing, which should lead to a **subcritical (hard) Hopf bifurcation.**

Figure 3.1: Hopf bifurcation using a computer #02. Phase portrait for the system in (3.1) when $\mu = 0.1 > 0$.

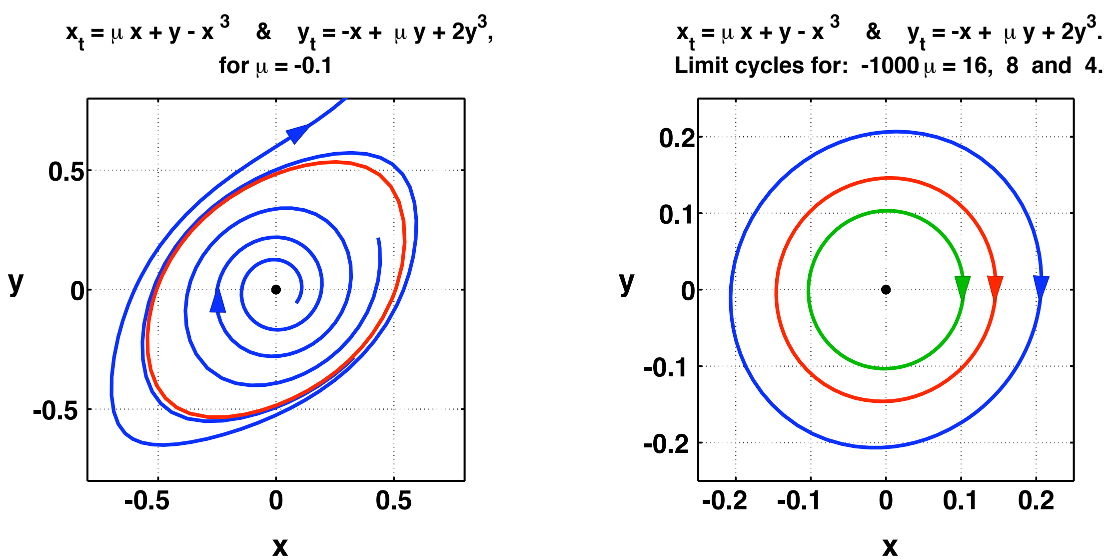


Figure 3.2: Hopf bifurcation using a computer #02. Left: phase portrait for equation (3.1) for $\mu = -0.1 < 0$. Right: the limit cycles for three values of $0 < -\mu \ll 1$, larger values of $|\mu|$ correspond to larger limit cycles.

The limit cycle for $\mu = -0.1$ is not very circular, but if we interpret its radius as the value of x when $y = 0$, we obtain $R \approx 0.5$, which yields $\frac{R}{\sqrt{|\mu|}} = 1.6$ (quite close to the values above in the table). In this case the period is $P \approx 2.1511\pi$.

4 Multiple scales and limit cycles #02

4.1 Statement: Multiple scales and limit cycles #02

Consider the nonlinear oscillator described by the equation

$$\frac{d^2x}{dt^2} - \epsilon\,(1-x^2)\,\frac{dx}{dt} + \frac{1}{\epsilon}\,\sin(\epsilon\,x) = 0, \tag{4.1}$$

where $0 < \epsilon \ll 1$. This system has a limit cycle: compute its approximate amplitude and period using the Poincaré-Lindstedt technique. Make sure that you compute at least the first nonvanishing correction to the linearized frequency. *Hint. There is a way to do this problem that requires a lot less algebra than a compute-blindly approach. Check the notes "Weakly Nonlinear Things: Oscillators", and use the fact that the argument of the sine in the equation is small.*

4.2 Answer: Multiple scales and limit cycles #02.

We can write equation (4.1) in the form

$$\frac{d^2x}{dt^2} - \epsilon\,(1-x^2)\,\frac{dx}{dt} + x = \frac{1}{6}\,\epsilon^2\,x^3 + O(\epsilon^4\,x^5). \tag{4.2}$$

Note that this is, up to the $O(\epsilon^2)$ terms on the right, the same as the van der Pol equation treated[3] in section 1.2 of the *"Weakly Nonlinear Things: Oscillators"* notes in the course WEB page. Thus, the algebra there (setting $\nu = 1$), up to the point where the terms on the right in (4.2) "kick in", applies! The first change occurs at the level of equation (1.22) in the notes, where the term

$$\frac{1}{6}\,x_0^3 = \frac{1}{6}\,a^3\,\cos^3(T) = \frac{1}{24}\,a^3\,\cos(3\,T) + \frac{1}{8}\,a^3\,\cos(T) \tag{4.3}$$

must be added to the right hand side of the equation. This changes the conclusions in equation (1.23) of the notes to:

1. $\omega_2 = -\frac{1}{256}\,a^4 - \frac{1}{16}\,a^2 = -\frac{5}{16}$.

2. $A = 0$.

3. $x_2 = \alpha\,\cos(T) + \frac{3}{512}\,a^3\,(2-a^2)\,\cos(3\,T) - \frac{5}{3072}\,a^5\,\cos(5\,T) - \frac{1}{192}\,a^3\,\cos(3\,T)$
$= \alpha\,\cos(T) - \frac{19}{192}\,\cos(3\,T) - \frac{5}{96}\,\cos(5\,T)$,

where α is a constant to be determined at the next order, and we have used that $\boldsymbol{a = 2}$. The expansion for the limit cycle, up to the order computed, is then

$$\begin{aligned} x &= 2\,\cos(T) - \frac{1}{4}\,\epsilon\,\sin(3\,T) \\ &\quad + \epsilon^2\left(\alpha\cos(T) - \frac{19}{192}\,\cos(3\,T) - \frac{5}{96}\,\cos(5\,T)\right) + O(\epsilon^3), && (4.4)\\ T &= \omega\,t, && (4.5)\\ \omega &= 1 - \frac{5}{16}\,\epsilon^2 + O(\epsilon^3), && (4.6) \end{aligned}$$

where α is still to be determined (must go to next order to do so).

[3] Using, precisely, the Poincaré-Lindstedt technique requested for this problem!

5 Multiple scales and limit cycles #03

5.1 Statement: Multiple scales and limit cycles #03

Consider the nonlinear oscillator described by the equation, for $\boldsymbol{x = x(t)}$,

$$\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} - \epsilon\,(1 - x^4)\,\frac{\mathrm{d}x}{\mathrm{d}t} + x = 0, \tag{5.1}$$

where $0 < \epsilon \ll 1$. This system has a limit cycle: compute its approximate amplitude and period using the method of multiple scales (also known as two-timing). The idea is to adapt the method presented in lectures for determining the amplitude of the limit cycle for the van der Pol oscillator.

Start by replacing the "single-time" dependence in $x = x(t)$ by a "two-times" dependence $\boldsymbol{x = x(t,\, \tau)}$, where $\boldsymbol{\tau = \epsilon\, t}$ is the "slow-time" over which the parameters in the harmonic oscillator ($\epsilon = 0$ case) solutions approximating the solutions to (5.1) evolve. Then

1. Then (from the chain rule) $\dfrac{\mathrm{d}x}{\mathrm{d}t} \mapsto \dfrac{\partial x}{\partial t} + \epsilon\,\dfrac{\partial x}{\partial \tau}$ and $\dfrac{\mathrm{d}^2 x}{\mathrm{d}t^2} \mapsto \dfrac{\partial^2 x}{\partial t^2} + 2\,\epsilon\,\dfrac{\partial^2 x}{\partial t\,\partial \tau} + \epsilon^2\,\dfrac{\partial^2 x}{\partial \tau^2}$ in (5.1).
2. Use the expansion $x = x_0(t,\, \tau) + \epsilon\, x_1(t,\, \tau) + O(\epsilon^2)$, to **show that** $$\big(\partial_{tt}\, x_0 + x_0\big) + \epsilon\big(\partial_{tt}\, x_1 + x_1 + 2\,\partial_{t\tau}\, x_0 + (x_0^4 - 1)\,\partial_t\, x_0\big) = O(\epsilon^2).$$
3. **Solve** the $O(1)$ system for x_0 real. Write your answer in terms of the complex amplitude $A(\tau)$.
4. Use your answer for x_0, and **compute the quantities** $2\,\partial_{t\tau}\, x_0$, x_0^4 and $\partial_t\, x_0$ in terms of A. By considering the secular terms that arise in the $O(\epsilon)$ system, **show that** $A(\tau)$ must satisfy $$A_\tau = \frac{1}{2}\,A - A\,|A|^4. \tag{5.2}$$
5. Write A in polar form $A = r(\tau)\, e^{i\,\phi(\tau)}$, and **find the differential equations** satisfied by r and ϕ. **Determine the stable fixed point** of the radial equation.
6. **Deduce that** the limit cycle solution is (approximately) given by $\boldsymbol{x = 2^{3/4}\cos(t) + O(\epsilon)}$.

5.2 Answer: Multiple scales and limit cycles #03.

2. The equation of motion is $$\big(\partial_{tt} + 2\,\epsilon\,\partial_{t\tau} + \epsilon^2\,\partial_{\tau\tau}\big)\,x + \epsilon\,(x^4 - 1)\,(\partial_t + \epsilon\,\partial_\tau)\,x + x = 0.$$ Substituting in the expansion for x and grouping together powers of ϵ gives the desired answer.
3. We have $\boldsymbol{x_0(t,\, \tau) = A(\tau)\, e^{i\,t} + A^*(\tau)\, e^{-i\,t}}$, where A^* denotes the complex conjugate of A.
4. In this part we write $z + \text{c.c.}$ to denote $z + z^*$, where the complex conjugation c.c. corresponds to the term that immediately precedes it. Then $$\begin{aligned} x_0^4 &= (A^4\, e^{4\,i\,t} + \text{c.c.}) + 4\,(A^2\,|A|^2\, e^{2\,i\,t} + \text{c.c.}) + 6\,|A|^4, \\ \partial_t\, x_0 &= i\,A\,e^{i\,t} + \text{c.c.}, \\ 2\,\partial_{t\tau}\, x_0 &= 2\,(i\,A_\tau\, e^{i\,t} + \text{c.c.}). \end{aligned}$$ The forcing term in the ode[4] for x_1 is $-(2\,\partial_{t\tau}\, x_0 + (x_0^4 - 1)\,\partial_t\, x_0)$. To identify secular terms, we need to compute the coefficients of $e^{i\,t}$ and $e^{-i\,t}$ (complex conjugates of each other). These follow easily from the calculation above. Setting these coefficients to zero yields $\boldsymbol{2\,i\,A_\tau + (6\,|A|^4 - 1)\,i\,A - 4\,i\,A\,|A|^4 = 0}$, which simplifies to give the desired result.

[4] The $O(\epsilon)$ equation.

5. Differentiating with respect to τ the polar form $A = r\,e^{i\,\phi}$ yields $A_\tau = (r_\tau + i\,r\,\phi_\tau)\,e^{i\,\phi}$. Substituting into (5.2), multiplying by $e^{-i\,\phi}$, and collecting real and imaginary parts, we obtain

$$r_\tau = \frac{r}{2} - r^5 \quad \text{and} \quad \phi_\tau = 0. \tag{5.3}$$

The fixed points for the radial equation are $\boldsymbol{r = 0}$ (unstable, corresponds to the origin in the original system) and $\boldsymbol{r = 2^{-1/4}}$ (stable, corresponds to a limit cycle for the original equation). Note that $\phi = \phi_0$ is a constant.

6. The behavior along the limit cycle is thus $\boldsymbol{x_0 = 2^{-1/4}\,(e^{i\,(t+\phi_0)} + e^{-i\,(t+\phi_0)}) = 2^{3/4}\cos(t+\phi_0)}$. Without loss of generality (choice of the time origin) we can set $\boldsymbol{\phi_0 = 0}$ to obtain the desired result.

THE END.