Answers to P-Set # 05, (18.353/12.006/2.050)j MIT (Fall 2024)

Rodolfo R. Rosales (MIT, Math. Dept., room 2-337, Cambridge, MA 02139)

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1.1 Statement: Attracting Lines in the Phase Plane #01

You may have observed [1] the frequent appearance of special curves along which solutions tend to bunch up. Sometimes these are associated with the stable and/or unstable manifolds of certain critical points, and sometimes they are not. Below is a particular example, where you are expected to justify the special line with a scaling/asymptotic kind of analysis (coupled with an appropriate qualitative argument on the phase plane). The problem is aimed at testing that you can use these sort of tools properly, so do it using them. If you think of another way of doing it, and you want to show off, then I will look at it and consider it for extra credit — but this can backfire if this "other way" contains some grave conceptual error (thus do it only if you are certain).

[1] Say, in the phase plane plots with the problem set answers, or if you played with the MatLab scripts in the course toolkit.

Consider the damped pendulum equation:

$$\frac{d^2\boldsymbol{\theta}}{dt^2} + a\,\frac{d\boldsymbol{\theta}}{dt} + \sin\boldsymbol{\theta} = \mathbf{0}, \qquad (1.1)$$

where $\boldsymbol{a} \gg \mathbf{1}$. **Show that ALL the trajectories** in the phase plane $(\boldsymbol{\theta},\, \dot{\boldsymbol{\theta}})$ end up, after a brief transient period, following the curve:

$$\frac{d\boldsymbol{\theta}}{dt} \approx -\frac{\mathbf{1}}{a}\,\sin\boldsymbol{\theta}. \qquad (1.2)$$

Hint. Do a phase portrait with a computer to see what is happening.

Remark 1.1 (Warnings)

a. *Do not fall into the trap of just showing that $\dot{\theta} + \frac{1}{a}\sin\theta$ is small!* Both terms here are small, so showing this provides no new information. What you must show is that this expression is much smaller than either term, for example: $\dot{\theta} + \frac{1}{a}\sin\theta = O(a^{-2})$. In fact, it is possible to show that $\dot{\theta} + \frac{1}{a}\sin\theta = O(a^{-3})$.

b. *Do not fall into the trap of arguing that, just because some derivative is multiplied by a small parameter, you can neglect it* — this is not always true.[2] The qualitative argument in the phase plane is the step that will allow you to justify something like this.

c. *Show that **all** the trajectories (eventually) satisfy (1.2), not just that there are some that do.*

1.2 Answer: Attracting Lines in the Phase Plane #01

The **key idea** in solving this problem is to **scale the variables properly** — notice that the equation is already non-dimensional.

A **large damping coefficient $\boldsymbol{a} \gg \mathbf{1}$** means that the system "does not like" having a large (or even $O(1)$) velocity. Any velocity that is not small to begin with, will be rapidly damped (leading to large accelerations) by the balance

$$\frac{d^2\theta}{dt^2} + a\frac{d\theta}{dt} = -\sin\theta = O(1). \qquad (1.3)$$

In other words, because $\sin\theta$ is always an $O(1)$ term, if any of the other terms is large, then both must be large and must cancel each other to leave an $O(1)$ remainder only. This balance implies damping on a time scale $\Delta t = O(a^{-1})$, which is thus the time scale for any transient period.

After the transient period described in the prior paragraph, the velocity will be small and the **motion rather slow**. It is this second stage the problem statement is asking us to look at. Thus it seems obvious that we should *re-scale time to look at the motion in the slow time scale.* Namely: **introduce $\boldsymbol{t = \tau/\delta}$, where $\boldsymbol{\tau}$ is the re-scaled time and $\mathbf{0} < \boldsymbol{\delta} \ll \mathbf{1}$** (with δ to be found). The idea is that the time scale over which things happen is $\Delta\tau = O(1)$ — equivalently: $\Delta t = O(\delta^{-1})$, so things happen *slowly.*

The question is now: **How do we determine $\boldsymbol{\delta}$**? Again, we look at equation (1.1) and ask in which ways can the various terms balance (each balance determines a possible time scale).

- The balance of the first two terms determines the damping time scale mentioned above. It is not important after the transient period.
- The balance of the first and last terms determines the time scale used when nondimensionalizing equation (1.1) — i.e.: it corresponds to $\Delta t = O(1)$ for significant changes. **This balance is inconsistent,** because: if the time derivatives are $O(1)$, then nothing can balance the middle term, which would then be the only large term in the equation.
- By elimination, the only possible balance outside the transient period is that of the second and last terms. This yields .... $\boldsymbol{\delta} = \dfrac{\mathbf{1}}{\boldsymbol{a}}$.

Thus, define $\boldsymbol{t = a\tau}$, so that the equation becomes

$$\frac{d^2\theta}{d\tau^2} + a^2\left(\frac{d\theta}{d\tau} + \sin\theta\right) = 0, \quad \text{or} \quad \left\{\begin{array}{rcl} \dfrac{d\theta}{d\tau} & = & u, \\ \dfrac{du}{d\tau} & = & -a^2\,(u + \sin\theta)\,. \end{array}\right. \qquad (1.4)$$

The behavior of this last system is easy to visualize in the phase plane (see figure 1.1). We have:

[2] Examples where this fails are wide-spread in applications. They are not a mathematical curiosity.

- **For $u > -\sin\theta$.** $\dfrac{du}{d\tau} \ll -1$ and $\dfrac{d\theta}{d\tau} = O(1)$. Orbits "zip" down towards the curve $u = -\sin\theta$ (nearly) vertically.
- **For $u < -\sin\theta$.** $\dfrac{du}{d\tau} \gg +1$ and $\dfrac{d\theta}{d\tau} = O(1)$. Orbits "zip" up towards the curve $u = -\sin\theta$ (nearly) vertically.

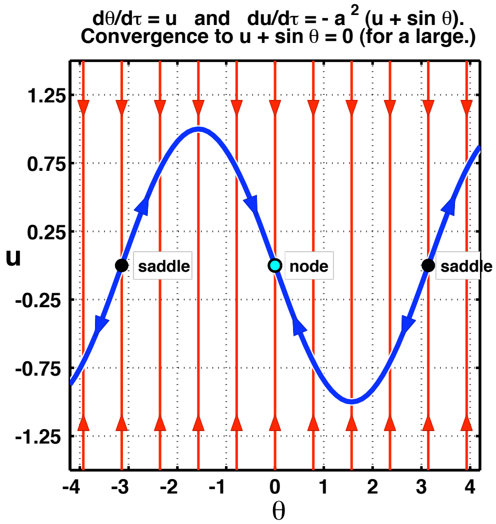

Figure 1.1: Sketch of the orbits for equation (1.4) in the phase plane, for $a \gg 1$. After a very brief transient period, the orbits approach the curve $u = -\sin\theta$, which they follow within a distance $O(a^{-2})$ — moving away from the saddles and approaching the nodes. The phase plane portrait is periodic in the θ direction, with period 2π.

Thus the orbits are very quickly attracted to a narrow region (of $O(a^{-2})$ thickness,[3] in this scaling), where they must stay, following the (approximately) 1-D evolution:

$$\frac{d\theta}{d\tau} = u = -\sin\theta + O(a^{-2}). \quad \text{Equivalently:} \quad \frac{d\theta}{dt} = -\frac{1}{a}\sin\theta + O(a^{-3}). \tag{1.5}$$

This is **exactly what the problem statement requires be shown.**

Remark 1.2 *Let us now show two other approaches to this problem, one of which falls into the pitfalls mentioned in remark* ***1.1****, while the other works.*

Other approach I — this does NOT work.

We can write equation (1.1) as the following system in the phase plane:

$$\frac{d\theta}{dt} = u, \qquad \text{and} \qquad \frac{du}{dt} = -a\left(u + \frac{1}{a}\sin\theta\right).$$

Then, using an argument similar to the one used above to conclude (1.5), we obtain

$$u + \frac{1}{a}\sin\theta = O(a^{-1}).$$

But **this is not good enough,** *as pointed out in remark* ***1.1****. The reason this approach does not work is the failure to do any re-scaling in the problem.* ♣

Other approach II — this works.

Introduce $v = a\,\frac{d\theta}{dt}$ and write equation (1.1) as the following system in the phase plane:

$$\frac{d\theta}{dt} = \frac{1}{a}v, \qquad \text{and} \qquad \frac{dv}{dt} = -a\,(v + \sin\theta).$$

Then, using an argument similar to the one used above to conclude (1.5), we obtain

$$v + \sin\theta = O(a^{-2}),$$

[3] In order for the "vertical" motion of the solutions in the $(u,\,\theta)$ phase plane (given by $du/d\tau$) not to dominate over the "horizontal" motion (given by $d\theta/d\tau$), it must be that $a^2\,(u + \sin\theta)$ is no larger than $O(1)$.

which is the same as (1.5). Notice that the reasoning leading to the last equation is: for the vertical motion not to dominate over the horizontal one (in the phase plane) it must be that $\dfrac{dv}{dt}$ *is no larger that* $O(a^{-1})$*, given the equation for* $\dfrac{d\theta}{dt}$. ♣

Remark 1.3 *The formula*

$$\frac{d^2\theta}{dt} \approx -\frac{1}{a}\sin\theta$$

that the solutions must satisfy (after a transient period) is an approximation, whose error is $O(a^{-3})$ *— as we pointed out earlier.* **Can we improve upon this?**
The answer is yes. *Generally, consider an equation of the form*

$$\frac{d^2x}{dt^2} + a\,\frac{dx}{dt} + f(x) = 0, \tag{1.6}$$

where $\boldsymbol{a \gg 1}$ *and* f *is* **smooth**. *Then the same arguments we used for (1.1) show that*

$$\frac{dx}{dt} = -\frac{1}{a}\,f(x) + O(a^{-3}). \tag{1.7}$$

Let us try the expansion:

$$\frac{dx}{dt} = -\frac{1}{a}\,f(x) + \frac{1}{a^3}\,g_1(x) + \frac{1}{a^5}\,g_2(x) + \dots \tag{1.8}$$

where g_1, g_2, *etc., are functions to be found. It is then clear that*

$$\begin{aligned} \frac{d^2x}{dt^2} &= \left(-\frac{1}{a}\,f'(x) + \frac{1}{a^3}\,g_1'(x) + \frac{1}{a^5}\,g_2'(x) + \dots\right)\frac{dx}{dt} \\ &= \left(\frac{1}{2\,a^2}\,f^2(x) - \frac{1}{a^4}\,f(x)\,g_1(x) + \frac{1}{a^6}\left(\frac{1}{2}\,g_1^2(x) - f(x)\,g_2(x)\right) + \dots\right)', \end{aligned} \tag{1.9}$$

where the primes denote derivatives with respect to x.
Now, substitute (1.8) and (1.9) into equation (1.6), and equate the coefficients of equal powers of a. *This yields*

$$\begin{aligned} g_1 &= -\left(\frac{1}{2}\,f^2\right)' = -f\,f', \\ g_2 &= (f\,g_1)' = -\left(f^2\,f'\right)' = -\left(\frac{1}{3}\,f^3\right)'', \\ g_3 &= \left(f\,g_2 - \frac{1}{2}\,g_1^2\right)', \end{aligned}$$

and so on. ♣

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2 Computer generated phase portrait: One Eye

2.1 Statement: Computer generated phase portrait: One Eye

A. Plot a computer generated phase plane portrait for the system $\dot{\boldsymbol{x}} = \boldsymbol{y} + \boldsymbol{y}^2$ and $\dot{\boldsymbol{y}} = -\boldsymbol{x} + \frac{1}{5}\,\boldsymbol{y} - \boldsymbol{x}\,\boldsymbol{y} + \frac{6}{5}\,\boldsymbol{y}^2$, (2.1)
in some "large" square, say: $-6 \le x,\ y \le 6$.
Note how many orbits eventually turn back towards the origin, to form the "eye".

B. Linearize the system near $(x,\ y) = (0,\ 0)$, and find **what type of critical point it is.**

C. Look at the phase portrait, and the result from **B**. **How are they consistent?**
Describe what happens near the critical point.

D. For $x \gg 1$, notice that the orbits approach a curve with $y \approx -1$. **Why does this happen?**

Hint for **C**. Do a detailed phase plane portrait near the eye.

Hint for **D**. The same type of approach used to analyze relaxation limit cycles works here, because something is large.

2.2 Answer: Computer generated phase portrait: One Eye

A. See figure 2.1.

B. The linearized system near the origin is $\dot{x} = y$ and $\dot{y} = -x + \frac{1}{5}\,y$, with eigenvalues $\lambda = \frac{1 \pm i\sqrt{99}}{10}$. Hence the critical point is an **unstable spiral**.

C. The panel on the left in figure 2.1 shows the trajectories approaching the eye, but **B** means that they cannot reach the critical point. The apparent contradiction is resolved by the existence of a **stable limit cycle** near the critical point — the "edge" of the eye. This limit cycle is shown on the right panel of figure 2.1.

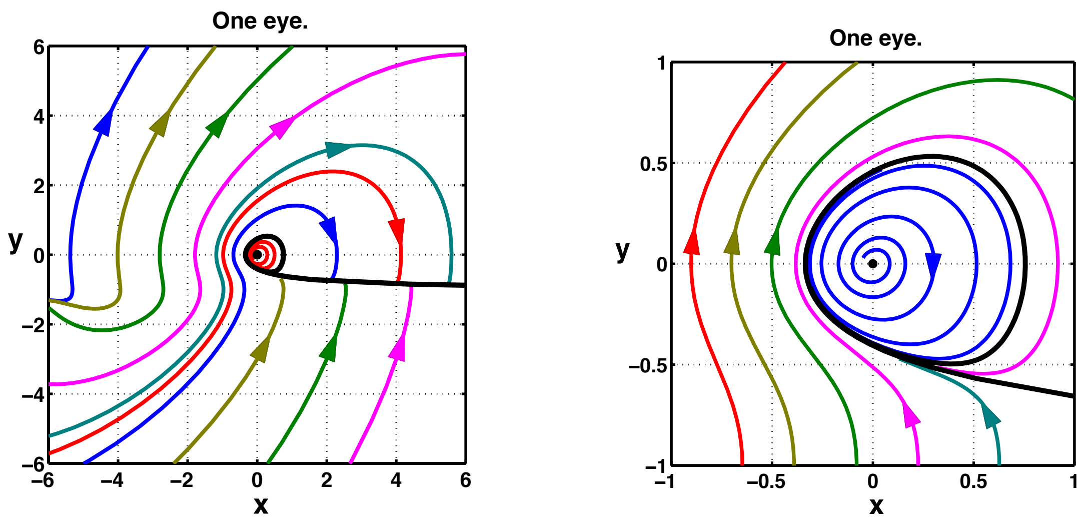


Figure 2.1: Phase plane portrait for (2.1). The picture on the right shows a detail near the limit cycle (the eye).

D. For $x \gg 1$ write $x = \frac{1}{\epsilon}\,\chi$, with $0 < \epsilon \ll 1$, $\chi = O(1)$. Hence $\dot{\chi} = \epsilon\,(y + y^2)$ and $\dot{y} = -\frac{1}{\epsilon}\,(y+1)\,\chi + \frac{1}{5}\,y + \frac{6}{5}\,y^2$. Thus, for $y > -1$, $\dot{y}$ is large and negative, while $\dot{\chi}$ is small. Similarly, for $y < -1$, $\dot{y}$ is large and positive, while $\dot{\chi}$ is small. **It follows that y gets pushed into a narrow band near $y = -1$.**

3 Computer generated phase portrait: van der Pol #02

3.1 Statement: Computer generated phase portrait: van der Pol #02

Task #1. Plot a computer generated phase plane portrait for the *van der Pol oscillator:*

$$\ddot{x} - 2\,(1 - x^2)\,\dot{x} + x = 0. \tag{3.1}$$

I strongly suggest that you use the PHPLdemoB *MatLab script provided to you in the class website [MatLab toolkit].* Note that, in order to use the script, you have to reduce the equation to a system written in terms of u and v; I suggest that you use $u = x$ and $v = \dot{x}$, and plot in the square $-5 < u, v < 5$ [this will give you a nice plot including the "main features" in the phase portrait].

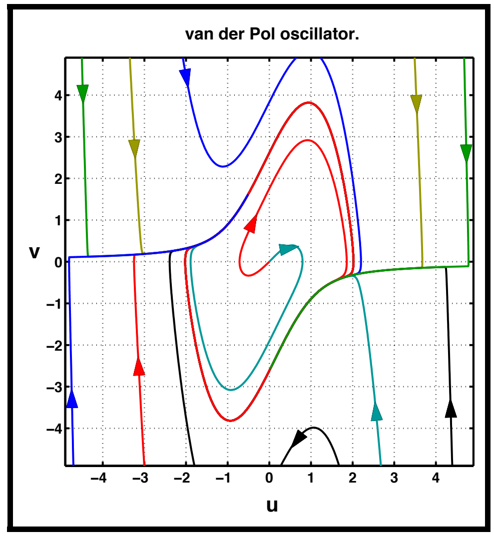
Task #2. What kind of critical point is $u = v = 0$? Find the eigenvalues of the linearized problem. Does your phase plane portrait agree with your analysis?

3.2 Answer: Computer generated phase portrait: van der Pol #02

With $u = x$ and $v = \dot{x}$ the equation in (3.1) becomes the system

$$\dot{u} = v \quad \text{and} \quad \dot{v} = -u + 2\,(1 - u^2)\,v.$$

A computer generated phase portrait for this system is shown by the picture on the right. Notice the **attracting limit cycle**. The origin is a **degenerate unstable node**, with a double eigenvalue $\lambda = 1$, and a single eigenvector $\vec{e} = (1,\,1)$. The direction of the eigenvector is evident in the plot, as the direction along which orbits leave the origin.



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4 Counter-rotating limit cycles

4.1 Statement: Counter-rotating limit cycles

Provide answers to the queries below.

(1) **True or false:** Is it possible to have a (smooth) phase plane system with **exactly two periodic orbits,** one of which lies inside the other, such that: *the inner orbit runs counterclockwise, and the outer orbit runs clockwise.* **If true:** do a sketch of a phase plane portrait with the stated property. **If false:** explain the reason. Note that **the orbits would be limit cycles,** because they would have to be isolated.
Hint #1. Beware of "gut feeling" instinctive answers. Your intuition may be faulty!
Hint #2. There is no loss of generality in assuming that, if such a system exists, the orbits are concentric circles. Then it helps to think in terms of polar coordinates. See remark below.

(2) If the answer to the first query is "true", what is the minimum number of critical points required?

Remark. *How can you be sure that a system written in polar coordinates is smooth?* Answer: write the system in the form

$$\dot{r} = a\,r \quad \text{and} \quad \dot{\theta} = b, \tag{4.1}$$

where a and b are some functions of $(r,\,\theta)$ — equivalently, of $(x,\,y)$. In cartesian coordinates this corresponds to

$$\dot{x} = a\,x - b\,y = f, \quad \dot{y} = b\,x + a\,y = g. \tag{4.2}$$

Then $a = a(x,\,y)$ and $b = b(x,\,y)$ should be such that f and g are smooth. **For example, this happens if $a = a(r^2)$ and $b = b(r^2)$ are smooth functions of r^2.** However, it generally fails if they are functions of $r = \sqrt{x^2 + y^2}$ only, because r is not smooth at the origin, which would render (4.2) not smooth there. ♣

Question: *why do we transform (4.1) to cartesian coordinates in order to determine if the system is smooth?* Answer: to remove the coordinate singularity at the origin, which interferes with the task.

4.2 Answer: Counter-rotating limit cycles

(1) **True.** Consider systems of the form (4.1), where $a = a(r^2)$ and $b = b(r^2)$, and select a and b so that

A. a has exactly two zeros, at $r^2 = a_1 > 0$ and $r^2 = a_2 > a_1$. Example: $a = (1 - r^2)(r^2 - 3)$.

B. b is such that $b(a_1) > 0$ and $b(a_2) < 0$. Example: $b = 2 - r^2$.

Any system with these properties provides an example with the required properties. Such a system has exactly two limit cycles (closed, isolated, orbits), that is: the circles $r \equiv \sqrt{a_1}$ (runs counterclockwise) and $r \equiv \sqrt{a_2}$ (runs clockwise). Note also that **the system has exactly one critical point:** the origin; a stable or unstable spiral depending on the sign of $a(0)$. No other point is a fixed point, since nowhere for $r > 0$ do both $\dot{r}$ and $\dot{\theta}$ vanish simultaneously. **Figure 4.1 shows the phase portrait for a system of this type.**

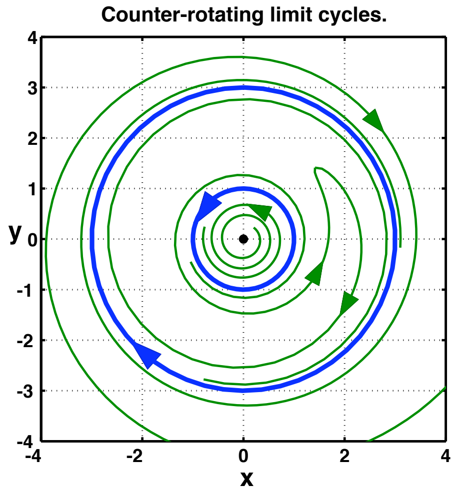

Figure 4.1: Counter-rotating limit cycles

The picture on the left shows a computer generated phase portrait for the system in (4.1–4.2), with

$$a = 0.03\,(1 - r^2)(r^2 - 9)/(3 + 2\,r^2)$$

and $$b = \arctan(0.3\,(4 - r^2)).$$

These particular forms were selected to get a plot where the structure is clearly visible (e.g.: avoid "crowded" spirals near the critical point). The two limit cycles, the critical point at the origin, and one typical orbit in each region are shown. The other orbits follow by rotation of these, since the system is invariant under rotation. The arrows indicate the flow direction.

(2) **The minimum number of critical points is one:** Index theory requires at least one critical point inside a limit cycle, and the example in figure 4.1 shows that this is enough.

5 Trapping regions; true or false

5.1 Statement: Trapping regions; true or false

True or false? There exists a phase plane system

$$\frac{dx}{dt} = f(x,\, y) \quad \text{and} \quad \frac{dy}{dt} = g(x,\, y), \qquad (5.1)$$

where f and g are fairly smooth (say, second partial derivatives continuous), and a simply connected (not empty) closed and bounded region $\mathcal{R}$, such that

A. $\mathcal{R}$ is a trapping region for the system in (5.1).

B. The system in (5.1) has no critical points in $\mathcal{R}$.

If true, give an example. If false, prove it. *Hint. Index theory and Poincaré Bendixon may help.*

An open set is simply connected if it has no holes. The technical definition is: *any loop (closed continuous curve) contained in the set can be continuously deformed into a point, while remaining inside the set.*

5.2 Answer: Trapping regions; true or false

False. We show this *by contradiction*: assume that the statement is true. Then

1. Take any orbit in $\mathcal{R}$. From the Poincaré Bendixon theorem, this orbit approaches a periodic orbit in $\mathcal{R}$. This because there are no critical points in $\mathcal{R}$, and $\mathcal{R}$ is bounded — hence all but one of the four choices† in the theorem are ruled out. Thus there is a periodic orbit inside $\mathcal{R}$.
2. From index theory, any periodic orbit must have (at least) one critical point inside. Since $\mathcal{R}$ is simply connected, there is a critical point in $\mathcal{R}$, which contradicts the assumptions.

† The choices are to approach: (1) A critical point. (2) Infinity. (3) A periodic orbit. (4) A cycle graph.

6 Volume evolution

6.1 Statement: Volume evolution

Consider some arbitrary orbit, $\mathbf{\Gamma}$, for the system

$$\frac{d\vec{r}}{dt} = \vec{F}(\vec{r}), \quad \text{where } \vec{r} \text{ and } \vec{F} \text{ are vectors in } \mathcal{R}^n, \tag{6.1}$$

and $\vec{F}$ has continuous partial derivatives up to (at least) second order. That is: $\mathbf{\Gamma}$ is the curve in $\mathcal{R}^n$ given by some solution $\boldsymbol{\vec{r} = \vec{r}_\gamma(t)}$ to (6.1). Then

A. Let $\Omega = \Omega(t)$ be an "infinitesimal" region that is being advected, along Γ, by the flow given by (6.1). *For example:*

A1. Let $\Omega(0)$ be a ball of "infinitesimal" radius dr, centered at $\vec{r}_\gamma(0)$.

A2. For every point $\vec{r}_p^{\,0} \in \Omega(0)$, let $\vec{r} = \vec{r}_p(t)$ be the solution to (6.1) defined by the initial data $\vec{r}_p(0) = \vec{r}_p^{\,0}$.

A3. At any time t_*, the set $\Omega(t_*)$ is given by all the points $\vec{r}_p(t_*)$, where $\vec{r}_p^{\,0}$ runs over all the points in $\Omega(0)$.

Note that $\Omega(0)$ need not be a ball. Any infinitesimal region containing $\vec{r}_\gamma(0)$ will do. All we need is that the notion of hypervolume applies to it — see item **B**. In particular: *you do not need to use/know the formula for the hypervolume of a ball in n dimensions to do this problem!*

B. Let $\mathcal{A} = \mathcal{A}(t)$ be the hypervolume of $\Omega(t)$. Note: (i) if $n = 1$ the hypervolume is the length; (ii) if $n = 2$ the hypervolume is the area; (iii) if $n = 3$ the hypervolume is the volume; etc.

TASK #1. Find a differential equation for the time evolution of $\mathcal{A}$.

TASK #2. Optional. Use the differential equation that $\mathcal{A}$ satisfies to show that $\det(e^{B\,t}) = e^{\text{tr}(B)\,t}$ for any square matrix B, where tr(B) denotes the trace of B. Note that you are required to do the proof using the differential equation, *specifically,* **not** by some other techniqe, like (say) linear algebra.

TASK #3. Optional. Prove the formula in (6.2) below.

Hints.

h1. Introduce the vector $\boldsymbol{\delta\vec{r}_p = \delta\vec{r}_p(t) = \vec{r}_p - \vec{r}_\gamma}$ for every point in $\Omega(t)$. This vector characterizes the evolution of the "shape" of Ω as the set moves along Γ. In order to calculate how $\mathcal{A}(t)$ evolves, you only need to know how the $\delta\vec{r}_p$ vectors evolve.

h2. For every vector $\delta\vec{r}_p$, write an equation giving $\delta\vec{r}_p(t + dt)$ in terms of $\delta\vec{r}_p(t)$ and the partial derivatives of $\vec{F}$ along Γ. Since you are dealing with infinitesimal terms, you can neglect higher order terms, so as to obtain a relationship from $\delta\vec{r}_p(t)$ to $\delta\vec{r}_p(t + dt)$ given by a linear transformation. Make sure that this linear transformation correctly includes the $O(dt)$ terms, which you will need to calculate time derivatives.

h3. From the transformation in item **h2** derive a relationship between $\mathcal{A}(t + dt)$ and $\mathcal{A}(t)$. Note that:
(a) For linear transformations, hypervolumes are related by the absolute value of the determinant. †
(b) You need to calculate the determinant only up to $O(dt)$ terms (neglect higher orders).
(c) For any square matrix $\boldsymbol{M}$,

$$\boldsymbol{\det(1 + \epsilon\, M) = 1 + \epsilon\, \text{tr}(M) + O(\epsilon^2)}. \tag{6.2}$$

h4. Item **h3** yields a formula of the form $\mathcal{A}(t + dt) = \mathcal{A}(t) + (\text{something})\, dt$.
Use this to get the differential equation for $\boldsymbol{\mathcal{A}}$.

† Multi-variable calculus: For a transformation $\vec{x} \to \vec{y}$; $\int f(\vec{y})\, d\vec{y} = \int f(\vec{y}(\vec{x}))\,|\det(J)|\, d\vec{x}$, where J = matrix of partial derivatives $\frac{\partial y_m}{\partial x_n}$. Thus for an infinitesimal hypervolume $\delta V \to |\det J|\, \delta V$. If $\vec{y} = M\,\vec{x}$ (linear transformation), $J = M$.

6.2 Answer: Volume evolution

At any time t, we can write

$$\vec{r}_\gamma(t+\mathrm{d}t) = \vec{r}_\gamma(t) + \vec{F}_\gamma(t)\,\mathrm{d}t, \tag{6.3}$$
$$\vec{r}_p(t+\mathrm{d}t) = \vec{r}_p(t) + \vec{F}_p(t)\,\mathrm{d}t, \tag{6.4}$$

where we have neglected $O\left((\mathrm{d}t)^2\right)$ contributions, $\vec{F}_\gamma = F(\vec{r}_\gamma)$, $\vec{F}_p = F(\vec{r}_p)$, and $\vec{r}_p = \vec{r}_p(t)$ tracks an arbitrary point in $\Omega(t)$ — as in item **A2**. Hence we can write

$$\begin{aligned}\delta\vec{r}_p\,(t+\mathrm{d}t) &= \delta\vec{r}_p\,(t) + \left(\vec{F}_p(t) - \vec{F}_\gamma(t)\right)\,\mathrm{d}t \\ &= \left(I + M_\gamma(t)\,\mathrm{d}t\right)\,\delta\vec{r}_p\,(t),\end{aligned} \tag{6.5}$$

where: **(a)** $\delta\vec{r}_p = \vec{r}_p - \vec{r}_\gamma$, **(b)** I is the identity matrix, **(c)** $M = \{\partial F_i/\partial x_j\}$ is the matrix of partial derivatives of $\vec{F}$, **(d)** $M_\gamma = M(\vec{r}_\gamma)$, and **(d)** we have neglected $O\left((dr)^2\,\mathrm{d}t\right)$ terms to arrive at the second line in (6.5). Therefore

$$\mathcal{A}(t+\mathrm{d}t) = \det\left(I + M_\gamma(t)\,\mathrm{d}t\right)\,\mathcal{A}(t) = \left(1 + \mathrm{tr}\left(M_\gamma(t)\right)\,\mathrm{d}t\right)\,\mathcal{A}(t), \tag{6.6}$$

where we have neglected $O\left((\mathrm{d}t)^2\right)$ terms when computing the determinant.[4] From this last equation we obtain

$$\frac{\mathbf{d}}{\mathbf{d}t}\,\boldsymbol{\mathcal{A}} = \mathbf{div}\left(\vec{\boldsymbol{F}}\right)\,\boldsymbol{\mathcal{A}}, \tag{6.7}$$

since $\mathrm{tr}\left(M_\gamma(t)\right) = \mathrm{div}\left(\vec{F}\right)$, **with the divergence evaluated along $\boldsymbol{\Gamma}$.**

The optional tasks

Consider the equation $\dfrac{\mathrm{d}\vec{r}}{\mathrm{d}t} = B\,\vec{r}$, for which (6.7) yields $\boldsymbol{\mathcal{A}} = e^{\mathbf{tr}(B)\,t}$ **[#1].** On the other hand $\vec{r}_p(t) = e^{B\,t}\,\vec{r}_p^{\,0}$, so that $\mathcal{A}(t) = \left|\det\left(e^{B\,t}\right)\right|$. But $\det\left(e^{B\,t}\right)$ is never zero and starts at one, hence $\det\left(e^{B\,t}\right) > 0$ and the absolute value is not needed. Thus $\boldsymbol{\mathcal{A}(t)} = \mathbf{det}\left(e^{B\,t}\right)$ **[#2].**
From [#1] and [#2], $\mathbf{det}(e^{B\,t}) = e^{\mathbf{tr}(B)\,t}$.

For the second optional task, consider first the case where all the eigenvalues of $\boldsymbol{M}$, $\{\boldsymbol{\lambda_n}\}$, are distinct, with $\boldsymbol{M\,\vec{v}_n = \lambda_n\,\vec{v}_n}$ the eigenvectors. It is then easy to see that the eigenvalues of $\mathbf{1} + \boldsymbol{\epsilon\,M}$ are $\{\mathbf{1} + \boldsymbol{\epsilon\,\lambda_n}\}$.
Hence: $\mathbf{det}(1 + \epsilon\,M) = \prod(1 + \epsilon\,\lambda_n) = 1 + \epsilon\,\sum\lambda_n + O(\epsilon^2) = 1 + \mathbf{tr}(M) + O(\epsilon^2)$.
The case of repeated eigenvalues follows by adding a small perturbation to $\boldsymbol{M}$, which splits the eigenvalues, and then taking the limit.
Note: **there are many other ways to prove this. For example:**
(a) You could expand the determinant using the first row, and then noticing that this yields $\mathbf{det}(1 + \epsilon\,M) = (1 + \epsilon\,m_{11})\,\mathbf{det}(1 + M_{11}) + O(\epsilon^2)$,‡ where $\boldsymbol{m_{ij}}$ are the entries of $\boldsymbol{M}$, and $\boldsymbol{M_{11}}$ is the matrix that results from eliminating the first row and column of $\boldsymbol{M}$. Then use induction.
‡ The tricky part here is showing that the $\boldsymbol{O(\epsilon^2)}$ is really $\boldsymbol{O(\epsilon^2)}$.
(b) You could use that, for any square matrix, $\mathbf{det}(A) = \sum \mathrm{sign}(\sigma_s)\,\Pi_s$, where the sum is over all the products $\boldsymbol{\Pi_s = \prod_i a_{i\,\sigma_s(i)}}$, where $\boldsymbol{\sigma_s}$ is a reordering of $\mathbf{1, 2, \ldots,}\ \boldsymbol{n}$. Then notice that, for $\boldsymbol{A} = \mathbf{1} + \boldsymbol{\epsilon\,M}$, the only $\boldsymbol{\sigma_s}$ for which $\boldsymbol{\Pi_s}$ is not $\boldsymbol{O(\epsilon^2)}$ (or higher) is the identity.
(c) You could use that[5] $\mathbf{ln\,det}(1 + \epsilon\,M) = \mathbf{tr}(\ln(1 + \epsilon\,M)) = \mathbf{tr}(\epsilon\,M - \frac{1}{2}\epsilon^2\,M^2 + \ldots)$.

THE END.

[4] No absolute value is needed because $I + M_\gamma(t)\,\mathrm{d}t$ is infinitesimally close to the identity, so its determinant is positive $= 1 + O(\mathrm{d}t)$.
[5] Note that $\mathbf{ln\,det}(1 + \epsilon\,M) = \mathbf{tr}(\ln(1 + \epsilon\,M))$ follows by taking the log of the result in the optional task #2.